Napoleonovi trokuti

...i još ponešto

Sonja Eberling

OŠ „Vladimir Gortan“, Rijeka

1. i 2. srpnja 2013.

Napoleon Bonaparte
(1769.–1821.)

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(1769.–1821.)

École Militaire

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(1769.–1821.)

École Militaire

Pierre-Simon Laplace
(1749.–1827.)

Lorenzo Mascheroni
(1750.–1800.)

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(1750.–1800.)

LA GEOMETRIA
DEL
COMPASSO
DI
LORENZO MASCHERONI.

PAVIA anno V della Repubblica Francese.

Presso gli Eredi di Pietro Galeazzi
(1797)

„La Geometria del Compasso“
(Pavia, 1797.)

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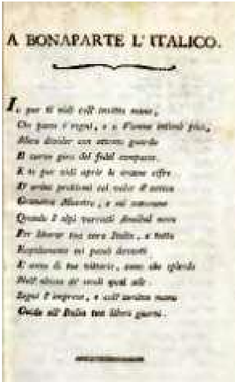

A BONAPARTE L' ITALICO.

posveta
„A Bonaparte l'Italico“

Sve konstrukcije koje se mogu izvesti ravnalom i šestarom mogu se izvesti i samo šestarom.

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EUCLIDES DANICUS,

Bestaaende udi Too Deele.

Forestillet.

Af

Georg Mohr.

„Euclides Danicus“
(Amsterdam, 1672.)

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EUCLIDES DANICUS,

Bestaande udi Too Deele.

Dend Förste Deel: Handler udaf de Sex Förste / EUCLIDIS Böger / ...

Dend Anden Deel: ...

Forestillet.
Af
Georg Mohr.

Prentet i Amsterdam af Jacob van Velsen. For Authore, Anno 1672.

„Euclides Danicus“
(Amsterdam, 1672.)

Johannes Trolle Hjelmslev
(1873.–1950.)
i
Julius Pál (Gyula Perl)
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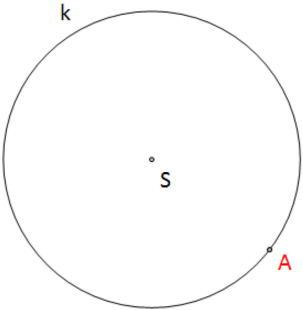
Mohr–Mascheronijeve konstrukcije

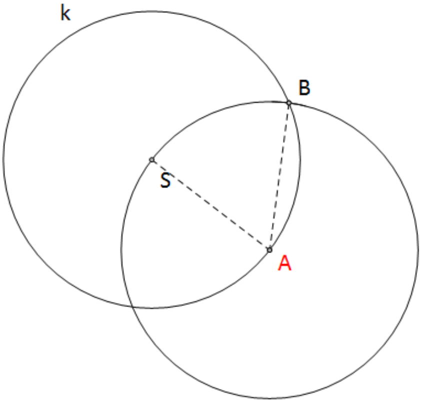
Napoleonov problem

Zadana je kružnica k sa središtem u točki S. Samo šestarom treba kružnicu podijeliti na 4 sukladna dijela.



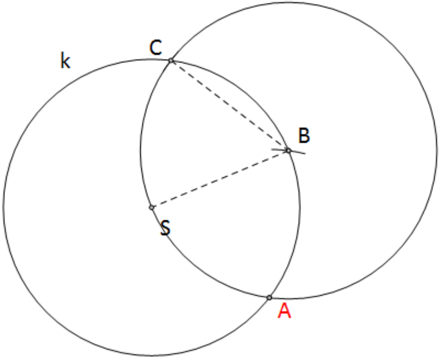
$$k = k(S, r)$$

$$A \in k$$



$$A \in k$$

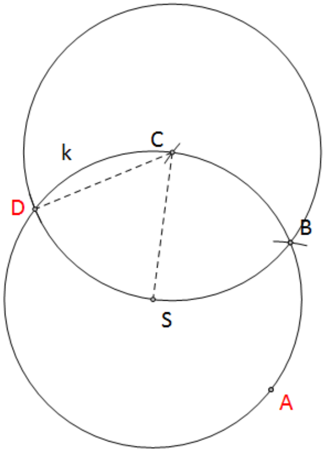
$$B \in k : |AS| = |BS| = r$$



$$A \in k$$

$$B \in k : |AS| = |BS| = r$$

$$C \in k : |BS| = |BC| = r$$

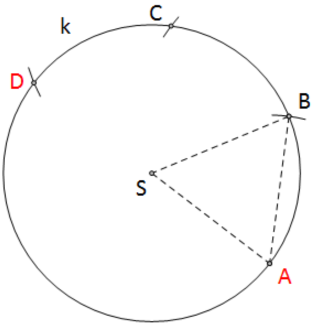


$$A \in k$$

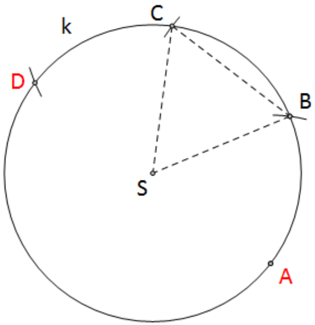
$$B \in k : |AS| = |BS| = r$$

$$C \in k : |BS| = |BC| = r$$

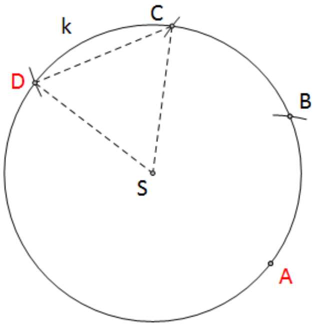
$$D \in k : |CS| = |CD| = r$$



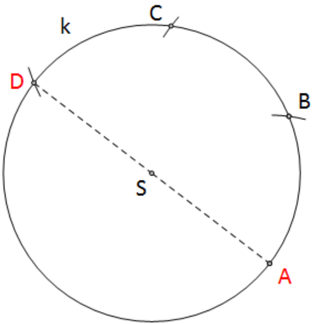
$\triangle SAB$ je jednakostraničan
$\Rightarrow |\measuredangle ASB| = 60^\circ$



$\triangle SAB$ je jednakostraničan
$\Rightarrow |\measuredangle ASB| = 60^\circ$
$\triangle SBC$ je jednakostraničan
$\Rightarrow |\measuredangle BSC| = 60^\circ$



$\triangle SAB$ je jednakostraničan
$\Rightarrow |\measuredangle ASB| = 60^\circ$
$\triangle SBC$ je jednakostraničan
$\Rightarrow |\measuredangle BSC| = 60^\circ$
$\triangle SCD$ je jednakostraničan
$\Rightarrow |\measuredangle CSD| = 60^\circ$



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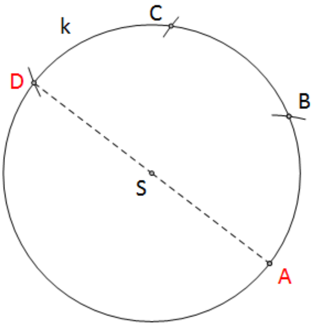
$\triangle SBC$ je jednakostraničan
$\Rightarrow |\measuredangle BSC| = 60^\circ$
$\triangle SCD$ je jednakostraničan
$\Rightarrow |\measuredangle CSD| = 60^\circ$

$$|\measuredangle ASD| = 180^\circ$$



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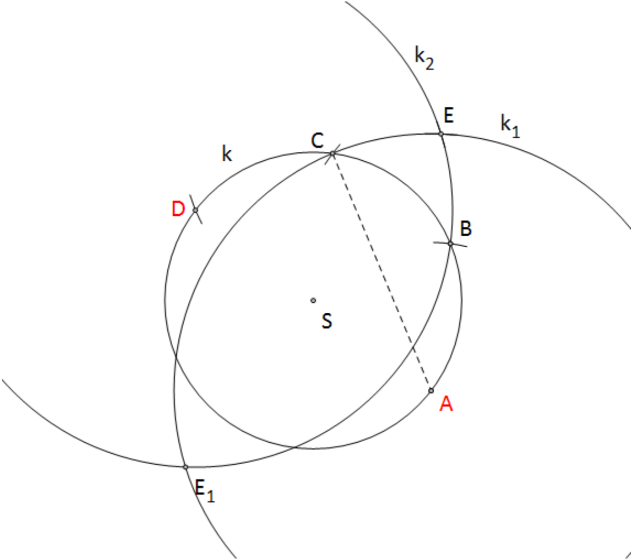
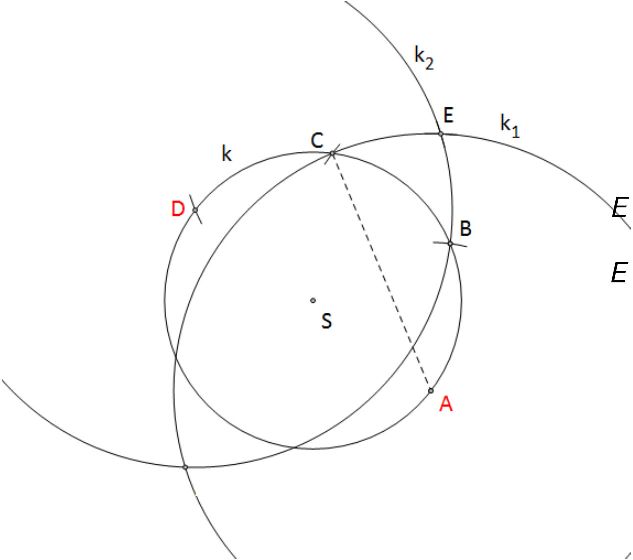
$\overline{AD}$ je promjer $k(S, r)$



$$k_1 = k(A, |AC|)$$

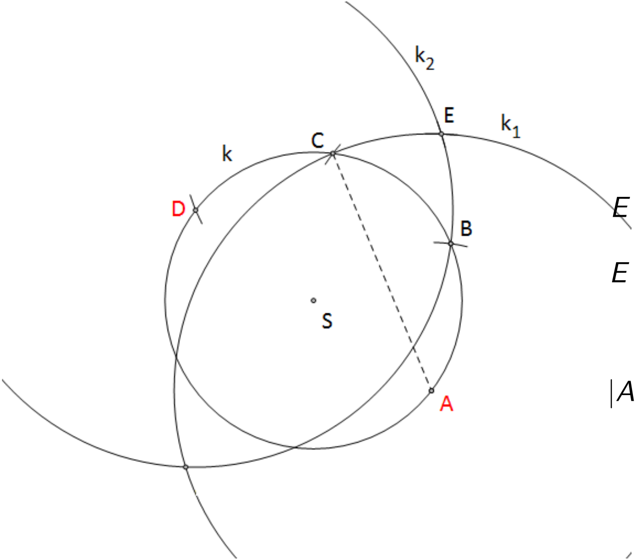
$$k_2 = k(D, |AC|)$$

$$k_1 \cap k_2 = \{E, E_1\}$$



$$E \in k_1 \Rightarrow |AC| = |AE|$$

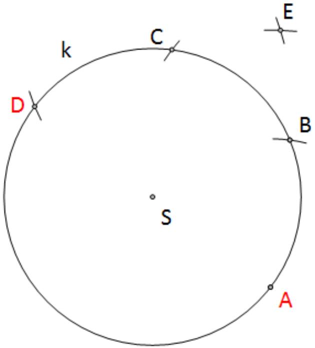
$$E \in k_2 \Rightarrow |AC| = |DE|$$

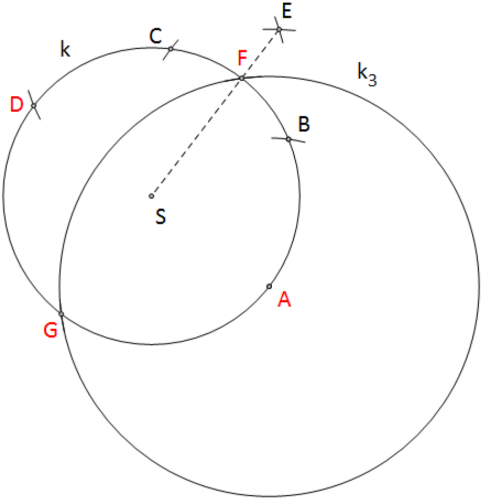


$$E \in k_1 \Rightarrow |AC| = |AE|$$

$$E \in k_2 \Rightarrow |AC| = |DE|$$

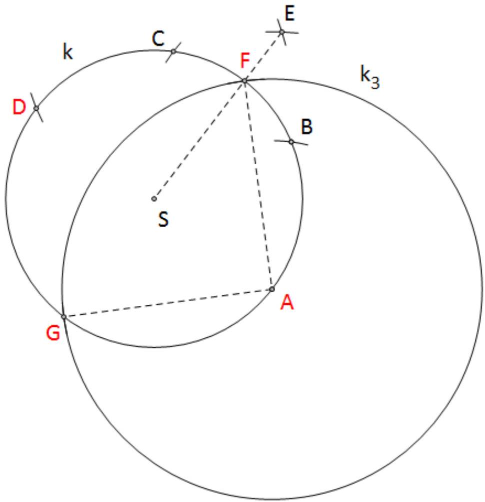
$$|AC| = |AE| = |DE| \qquad (1)$$


E
C
k
D
B
S
A



$$k_3 = k(A, |ES|)$$

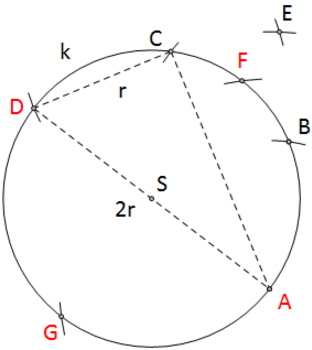
$$k_3 \cap k = \{F, G\}$$



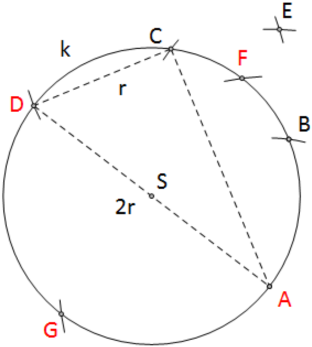
$$F \in k_3 \Rightarrow |ES| = |AF|$$

$$G \in k_3 \Rightarrow |ES| = |AG|$$

$$|ES| = |AF| = |AG| \qquad (2)$$



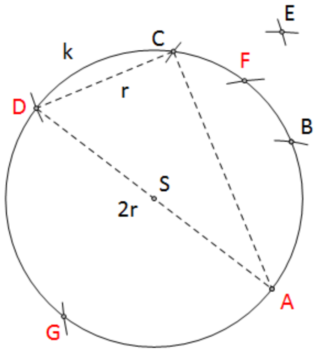
$\overline{AD}$ je promjer $k(S, r)$



$\overline{AD}$ je promjer $k(S, r)$

$\xrightarrow{\text{Talesov poučak}} |\measuredangle DCA| = 90^\circ$

$\Rightarrow \triangle ACD$ je pravokutan



$\overline{AD}$ je promjer $k(S, r)$

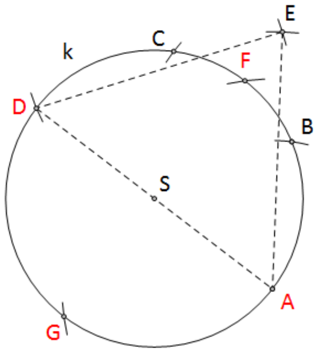
$\xRightarrow{\text{Talesov poučak}} |\measuredangle DCA| = 90^\circ$

$\Rightarrow \triangle ACD$ je pravokutan

$$\xRightarrow{\text{Pitagorin poučak}} (2r)^2 = r^2 + |AC|^2$$

$$|AC|^2 = 3r^2$$

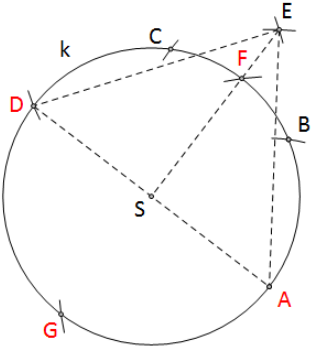
$$|AC| = r\sqrt{3} \tag{3}$$



$$\text{Iz } |AC| = |AE| = |DE| \qquad (1)$$
$$\text{i } \ |AC| = r\sqrt{3} \qquad (3)$$

$$\Rightarrow |AE| = |DE| = r\sqrt{3}$$

$\triangle AED$ je jednakokračan
($\overline{DA}$ je osnovica, S polovište osnovice, $\overline{ES}$ visina)



$$\text{Iz } |AC| = |AE| = |DE| \qquad (1)$$
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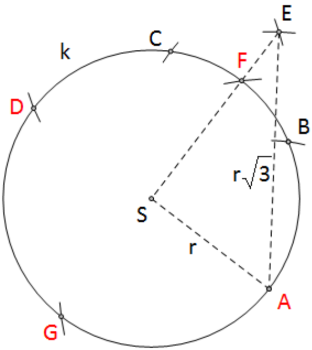
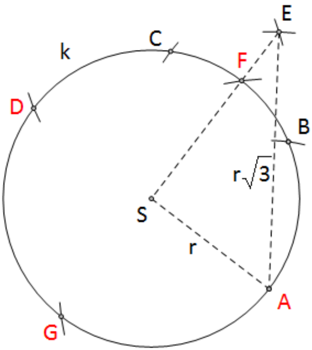
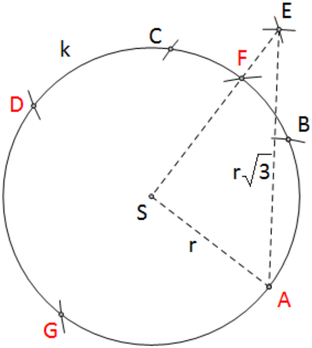
($\overline{DA}$ je osnovica, S polovište osnovice,
$\overline{ES}$ visina)

$\triangle ESA$ pravokutan


E
k
C
F
D
B
r√3
S
r
A
G



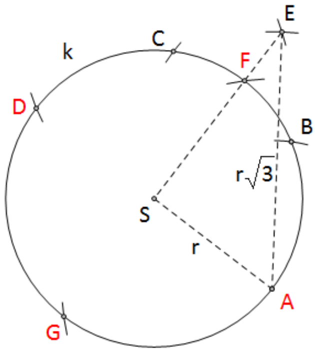
$$\xRightarrow{\text{Pitagorin poučak}} (r\sqrt{3})^2 = r^2 + |ES|^2$$



$$\xRightarrow{\text{Pitagorin poučak}} (r\sqrt{3})^2 = r^2 + |ES|^2$$

$$|ES|^2 = 2r^2$$

$$|ES| = r\sqrt{2}$$

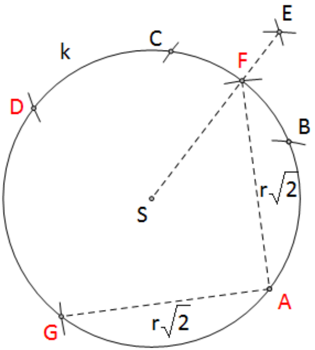


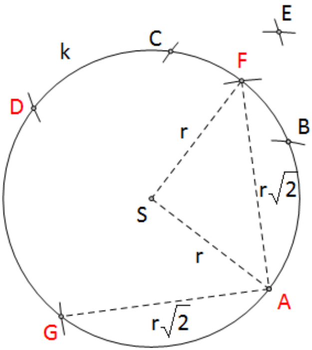
$$\xRightarrow{\text{Pitagorin poučak}} (r\sqrt{3})^2 = r^2 + |ES|^2$$

$$|ES|^2 = 2r^2$$

$$|ES| = r\sqrt{2}$$

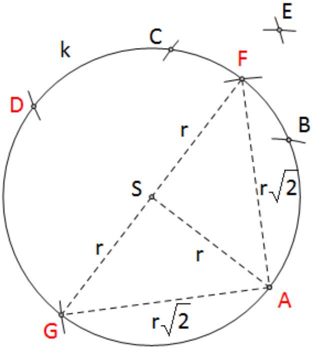
$$\xRightarrow{(2)} |AF| = |AG| = r\sqrt{2}$$


E
k
C
F
D
B
$r\sqrt{2}$
S
A
G
$r\sqrt{2}$



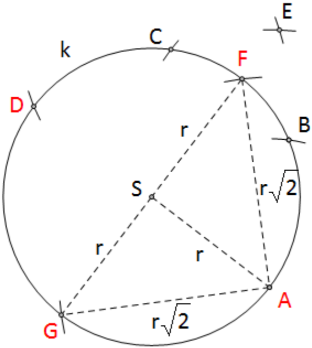
$$\xRightarrow{\text{obrat P.p.}} \triangle FSA \text{ je pravokutan}$$

$$|\measuredangle FSA| = 90^\circ$$



$\xRightarrow{\text{obrat P.p.}}$ $\triangle FSA$ je pravokutan

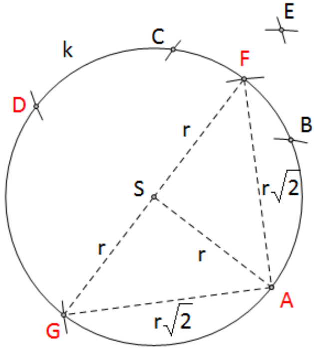
$$|\measuredangle FSA| = 90^\circ$$



$\xrightarrow{\text{obrat P.p.}}$ $\triangle FSA$ je pravokutan

$$|\measuredangle FSA| = 90^\circ$$

$$|\measuredangle GSA| = 90^\circ$$



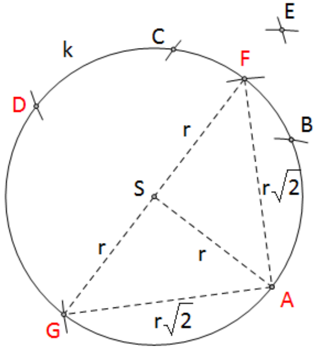
$$\xrightarrow{\text{obrat P.p.}} \triangle FSA \text{ je pravokutan}$$

$$|\measuredangle FSA| = 90^\circ$$

$$|\measuredangle GSA| = 90^\circ$$

$\overline{FG}$ je promjer $k(S, r)$

$$\overline{AD} \perp \overline{FG}$$



$$\xrightarrow{\text{obrat P.p.}} \triangle FSA \text{ je pravokutan}$$

$$|\measuredangle FSA| = 90^\circ$$

$$|\measuredangle GSA| = 90^\circ$$

$\overline{FG}$ je promjer $k(S, r)$

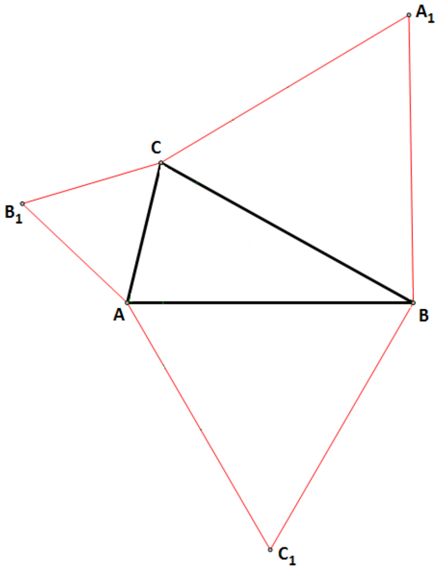
$$\overline{AD} \perp \overline{FG}$$

Točke A, F, D i G dijele kružnicu $k(S, r)$ na četiri sukladna dijela.

GSP

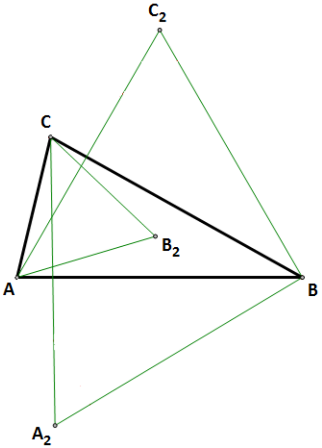
Nad stranicama trokuta ABC konstruiramo:

- „prema van“ jednakostranične trokute $\triangle AC_1B$, $\triangle BA_1C$, $\triangle CB_1A$



Nad stranicama trokuta ABC konstruiramo:

- „prema van" jednakostranične trokute $\triangle AC_1B$, $\triangle BA_1C$, $\triangle CB_1A$
- „prema unutra" jednakostranične trokute $\triangle ABC_2$, $\triangle BCA_2$, $\triangle CAB_2$.



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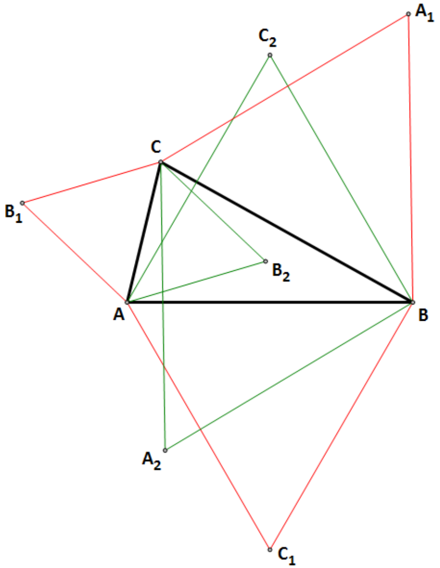


Uz ovako dobivenu figuru vezana su neka zanimljiva svojstva.

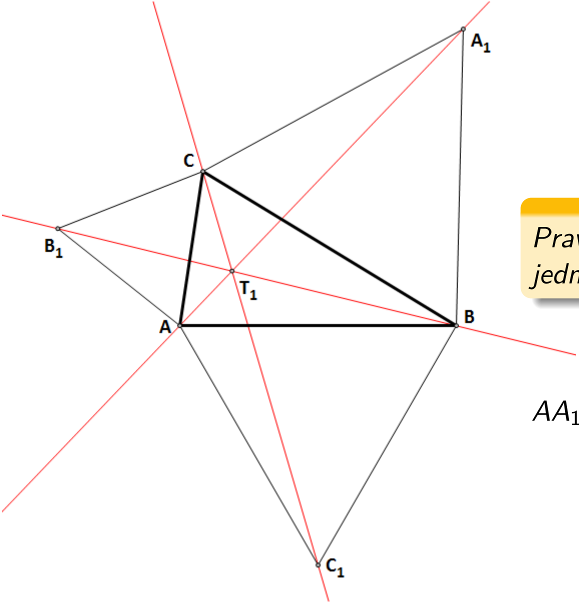
Giovanni Ceva
(1647.–1734.)

Tri pravca koji prolaze vrhovima trokuta i sijeku se u jednoj točki zovu se Cevaini pravci.

Giovanni Ceva
(1647.–1734.)

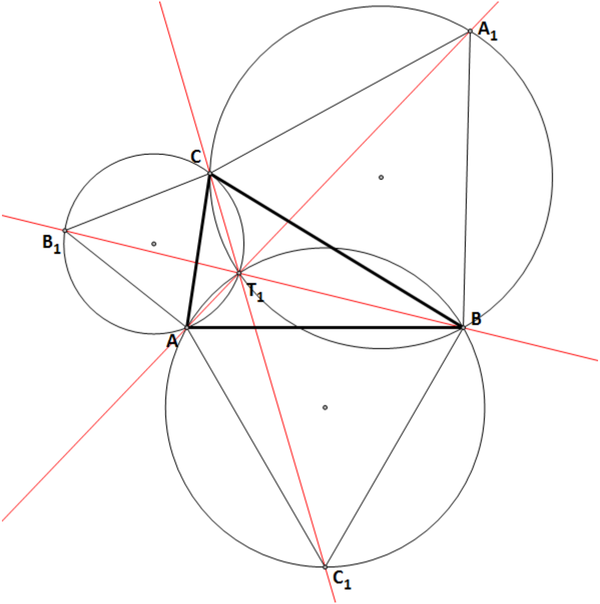
Tri pravca koji prolaze vrhovima trokuta i sijeku se u jednoj točki zovu se Cevaini pravci.

- simetrale unutarnjih kutova trokuta
- pravci kojima pripadaju visine trokuta
- pravci kojima pripadaju težišnice trokuta
- ...

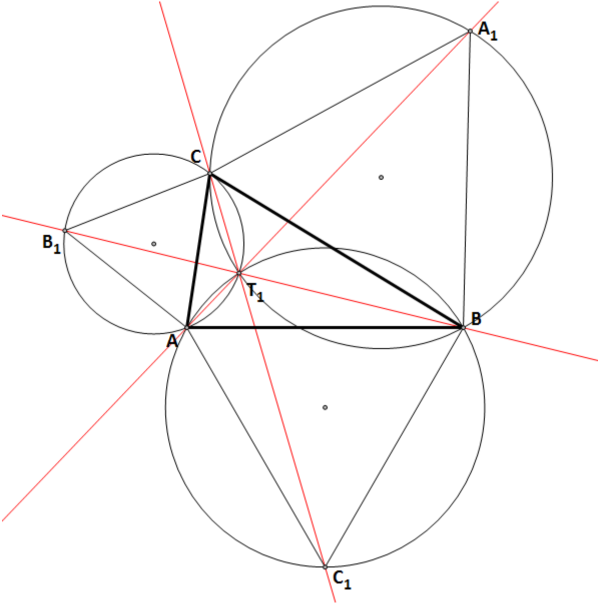


Pravci AA_1, BB_1 i CC_1 prolaze jednom točkom.

AA_1, BB_1, CC_1 – Cevaini pravci

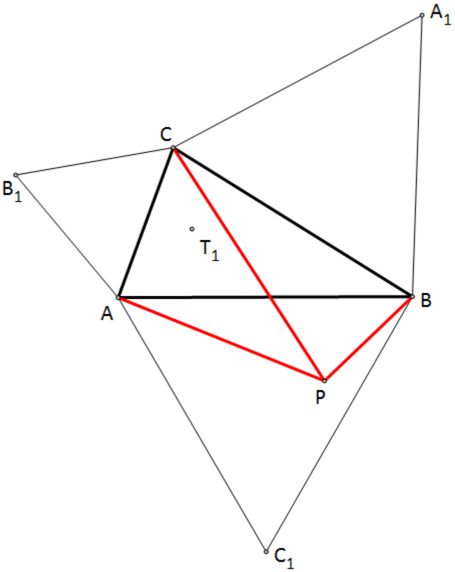


Kružnice opisane trokutima $\triangle AC_1B$, $\triangle BA_1C$ i $\triangle CB_1A$ prolaze jednom točkom.

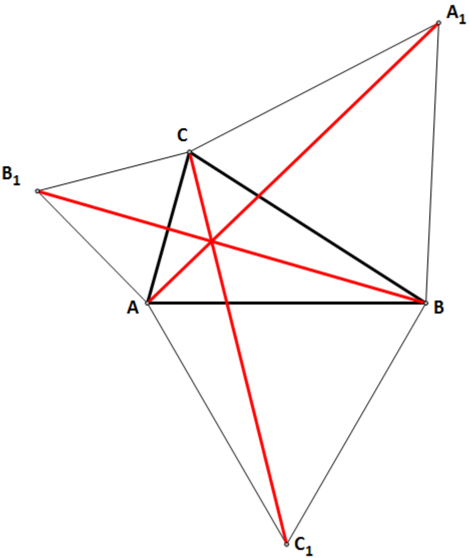


Kružnice opisane trokutima $\triangle AC_1B$, $\triangle BA_1C$ i $\triangle CB_1A$ prolaze jednom točkom.

- Torricellijeve kružnice
- T_1–Torricellijeva (Fermatova) točka trokuta

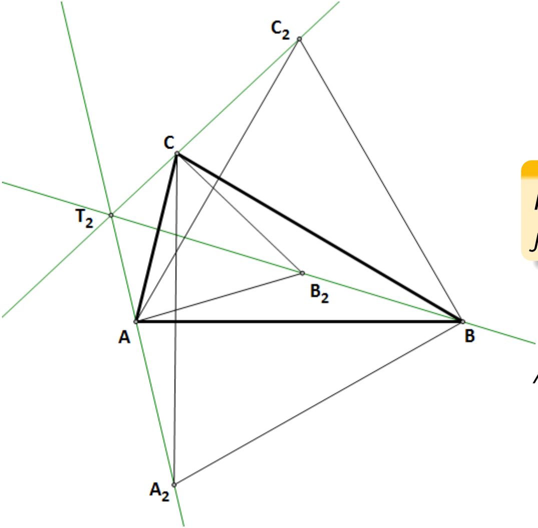


Zbroj udaljenosti neke točke P do vrhova $\triangle ABC$ je minimalan ako se točka P poklapa s točkom T_1.



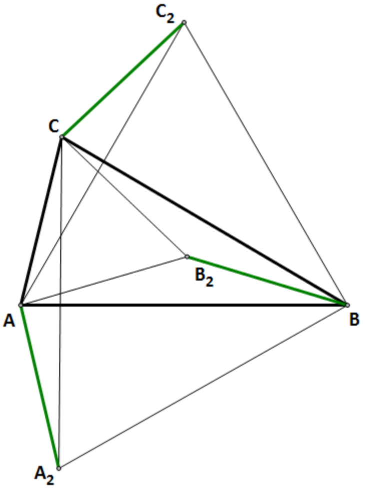
$$|AA_1| = |BB_1| = |CC_1|$$

GSP



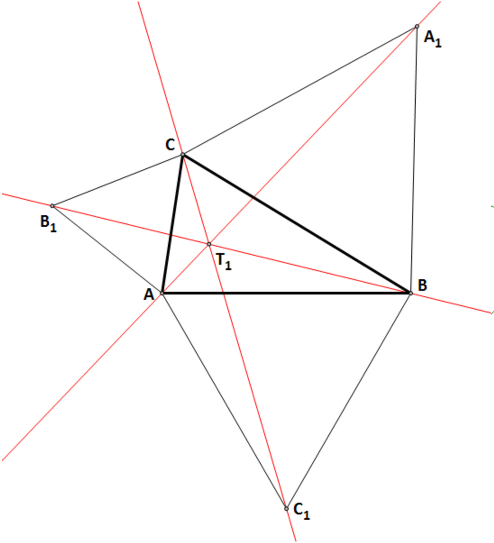
Pravci AA_2, BB_2 i CC_2 prolaze jednom točkom.

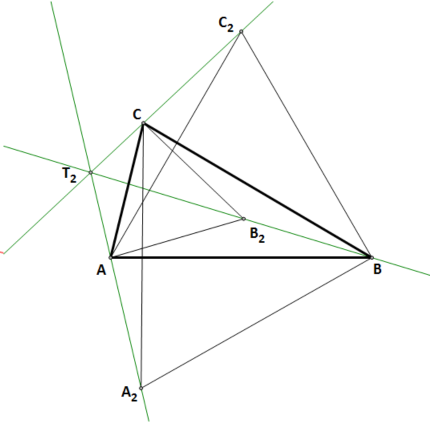
AA_2, BB_2, CC_2 – Cevaini pravci



$$|AA_2| = |BB_2| = |CC_2|$$

GSP

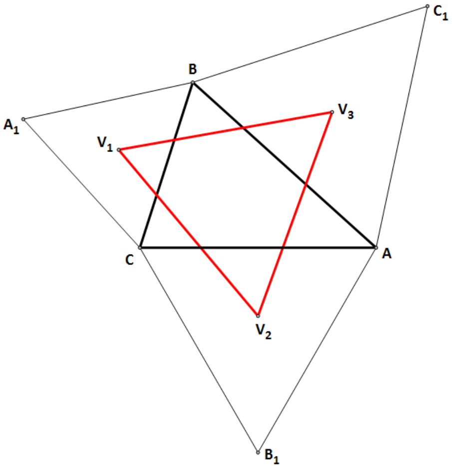




T_1 i T_2 – **izogonički centri** $\triangle ABC$

Napoleonov teorem

Ako nad stranicama trokuta s vanjske strane konstruiramo jednakostranične trokute, onda su središta tako dobivenih trokuta vrhovi jednakostraničnog trokuta.



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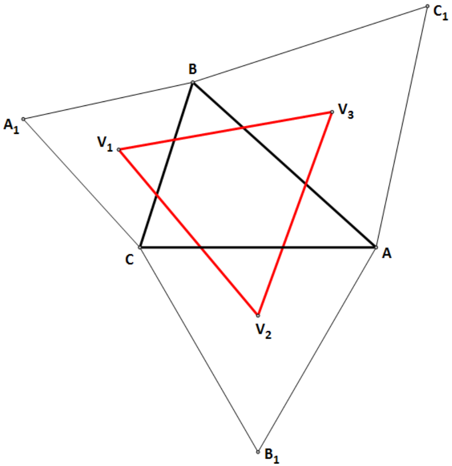
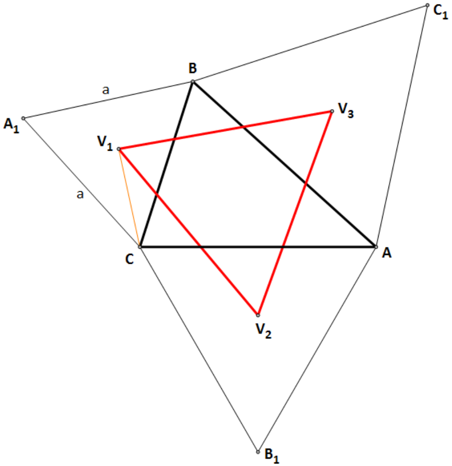
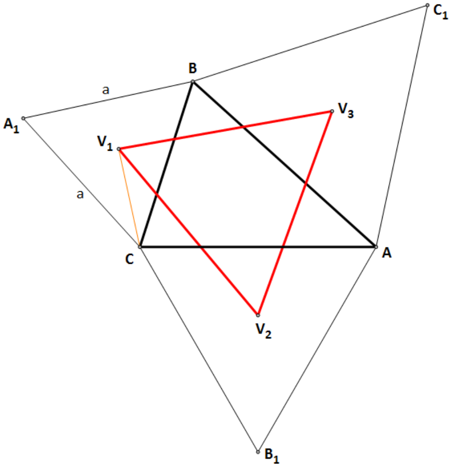


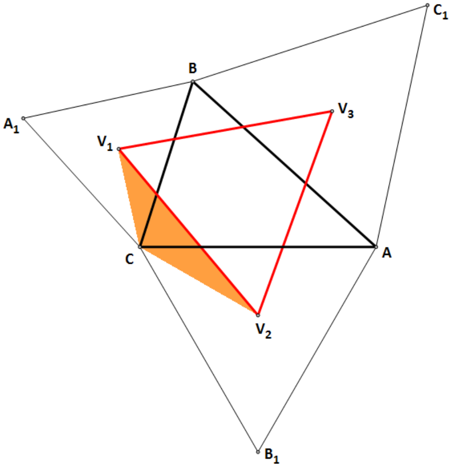
$\triangle V_1V_2V_3$ – **vanjski Napoleonov trokut** trokuta ABC


C1
B
a
V3
A1
V1
a
C
A
V2
B1

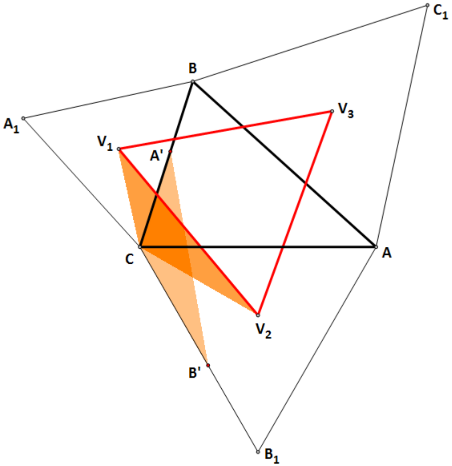


Neka su a, b, c duljine stranica $\triangle ABC$.

$$|CV_1| = \frac{2}{3} \cdot \frac{a\sqrt{3}}{2} = \frac{a\sqrt{3}}{3} = \frac{a}{\sqrt{3}}$$



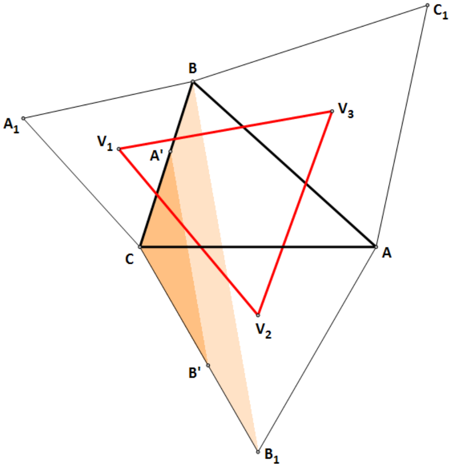
$$|CV_1| = \frac{a}{\sqrt{3}}$$



$$|CV_1| = \frac{a}{\sqrt{3}}$$

$$\triangle CV_2V_1 \xmapsto{r(C,-30^\circ)} \triangle CB'A'$$

$$|CV_1| = |CA'| = \frac{a}{\sqrt{3}}$$

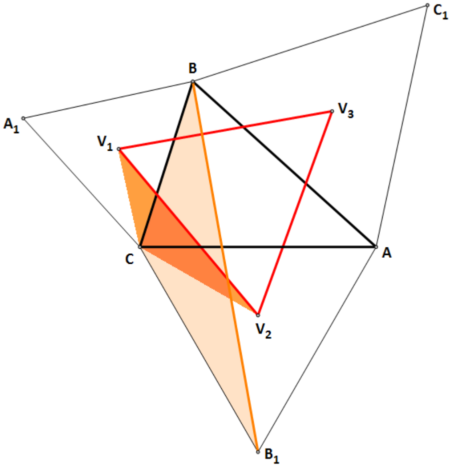


$$|CV_1| = \frac{a}{\sqrt{3}}$$

$$\triangle CV_2V_1 \xmapsto{r(C,-30^\circ)} \triangle CB'A'$$

$$|CV_1| = |CA'| = \frac{a}{\sqrt{3}}$$

$$\triangle CB'A' \xmapsto{h(C,\sqrt{3})} \triangle CB_1B$$



Neka je:

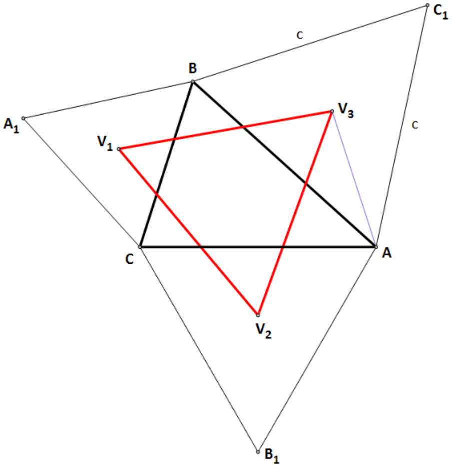
$$r_1 = r(C, -30^\circ)$$

$$h_1 = h(C, \sqrt{3})$$

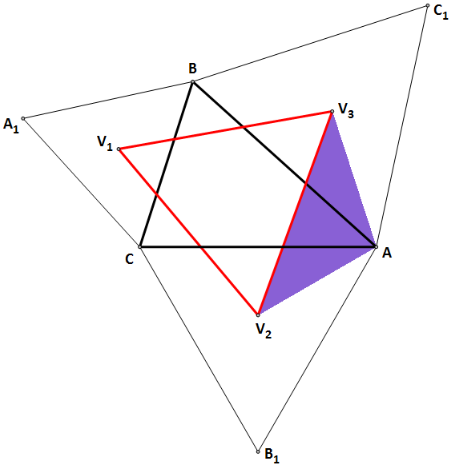
Tada je:

$$\overline{V_1V_2} \xmapsto{h_1 \circ r_1} \overline{BB_1}$$

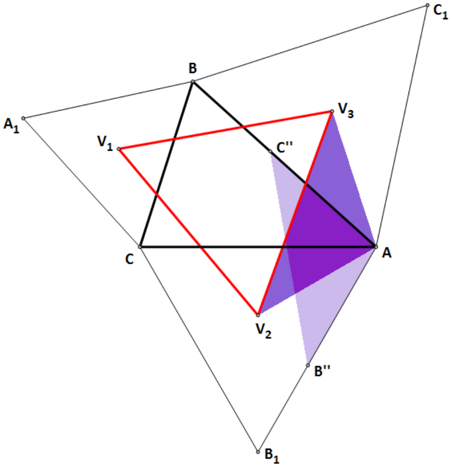
$$|V_1V_2| = \frac{|BB_1|}{\sqrt{3}} \tag{4}$$



$$|AV_3| = \frac{2}{3} \cdot \frac{c\sqrt{3}}{2} = \frac{c\sqrt{3}}{3} = \frac{c}{\sqrt{3}}$$



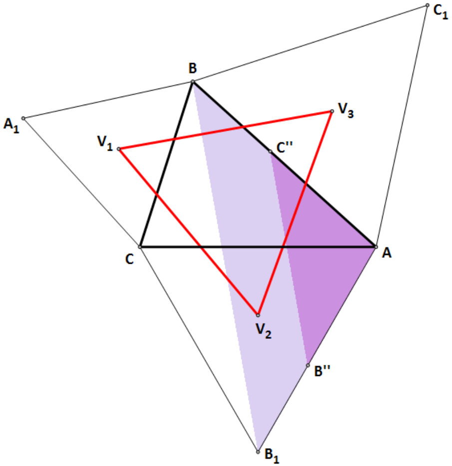
$$|AV_3| = \frac{c}{\sqrt{3}}$$



$$|AV_3| = \frac{c}{\sqrt{3}}$$

$$\triangle AV_3V_2 \xmapsto{r(A,30^\circ)} \triangle AC''B'$$

$$|AV_3| = |AC''| = \frac{c}{\sqrt{3}}$$

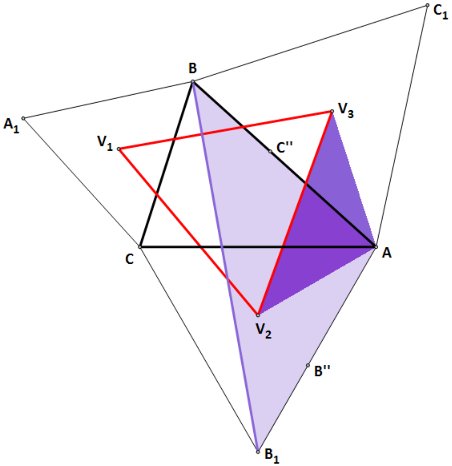


$$|AV_3| = \frac{c}{\sqrt{3}}$$

$$\triangle AV_3V_2 \xmapsto{r(A,30^\circ)} \triangle AC''B'$$

$$|AV_3| = |AC''| = \frac{c}{\sqrt{3}}$$

$$\triangle AC''B'' \xmapsto{h(A,\sqrt{3})} \triangle ABB_1$$



Neka je:

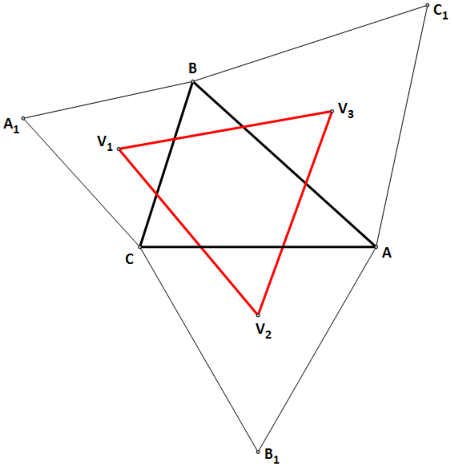
$$r_2 = r(A, 30^\circ)$$

$$h_2 = h(A, \sqrt{3})$$

. Tada je:

$$\overline{V_2V_3} \xmapsto{h_2 \circ r_2} \overline{B_1B}$$

$$|V_2V_3| = \frac{|BB_1|}{\sqrt{3}} \tag{5}$$



Iz (4) i (5) slijedi

$$|V_1 V_2| = |V_2 V_3|.$$

Analogno se pokazuje

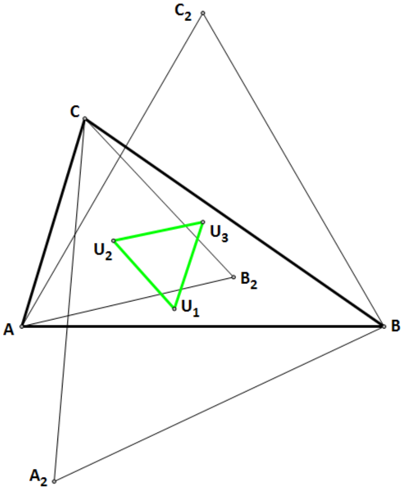
$$|V_2 V_3| = |V_1 V_3|.$$

Dakle,

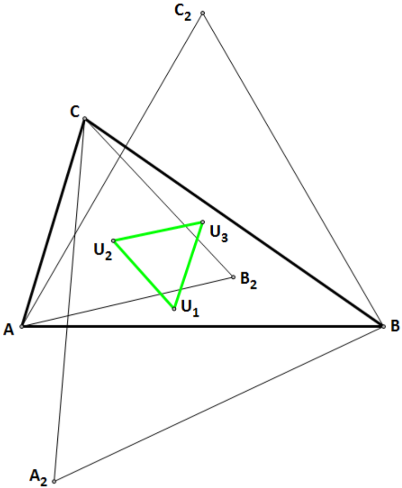
$$|V_1 V_2| = |V_2 V_3| = |V_1 V_3|$$

tj. $\triangle V_1 V_2 V_3$ je jednakostraničan.

Ako nad stranicama trokuta s unutarnje strane konstruiramo jednakostranične trokute, onda su središta tako dobivenih trokuta vrhovi jednakostraničnog trokuta.

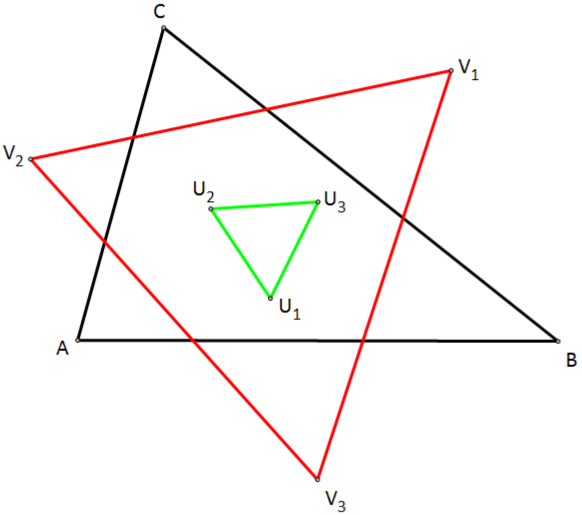


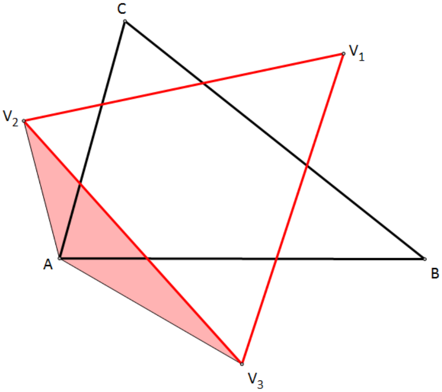
Ako nad stranicama trokuta s unutarnje strane konstruiramo jednakostranične trokute, onda su središta tako dobivenih trokuta vrhovi jednakostraničnog trokuta.

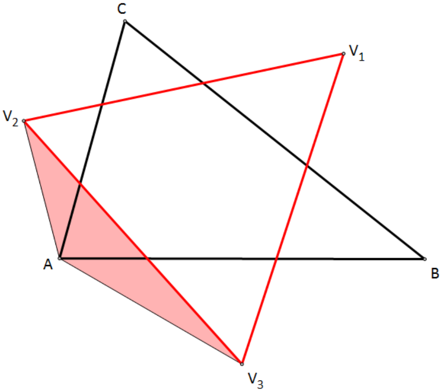


$\triangle U_1 U_3 U_2$ – **unutarnji Napoleonov trokut** trokuta ABC

Razlika površina vanjskog i unutarnjeg Napoleonovog trokuta trokuta ABC jednaka je površini trokuta ABC.

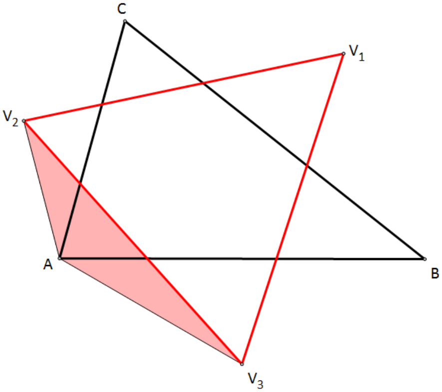



C
V1
V2
A
B
V3



Neka su a, b, c duljine stranica, a α, β, γ veličine unutarnjih kutova $\triangle ABC$.

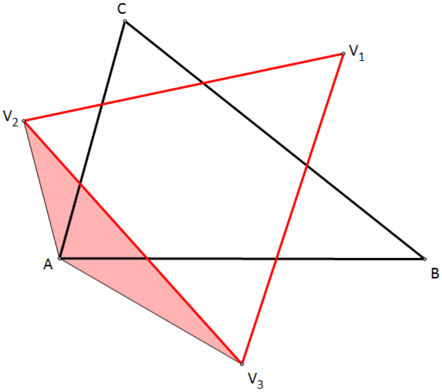
$$|\measuredangle V_2AV_3| = \alpha + 60^\circ$$



Neka su a, b, c duljine stranica, a α, β, γ veličine unutarnjih kutova $\triangle ABC$.

$$|\measuredangle V_2AV_3| = \alpha + 60^\circ$$

$$|AV_2| = \frac{b}{\sqrt{3}}$$

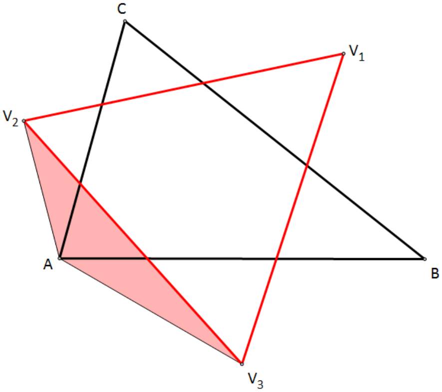


Neka su a, b, c duljine stranica, a α, β, γ veličine unutarnjih kutova $\triangle ABC$.

$$|\measuredangle V_2AV_3| = \alpha + 60^\circ$$

$$|AV_2| = \frac{b}{\sqrt{3}}$$

$$|AV_3| = \frac{c}{\sqrt{3}}$$



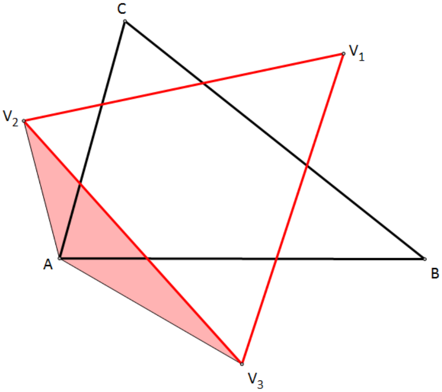
Neka su a, b, c duljine stranica, a α, β, γ veličine unutarnjih kutova $\triangle ABC$.

$$|\measuredangle V_2AV_3| = \alpha + 60^\circ$$

$$|AV_2| = \frac{b}{\sqrt{3}}$$

$$|AV_3| = \frac{c}{\sqrt{3}}$$

$$|V_2V_3|^2 = \frac{b^2}{3} + \frac{c^2}{3} - \frac{2bc}{3} \cdot \cos(\alpha + 60^\circ)$$



Neka su a, b, c duljine stranica, a α, β, γ veličine unutarnjih kutova $\triangle ABC$.

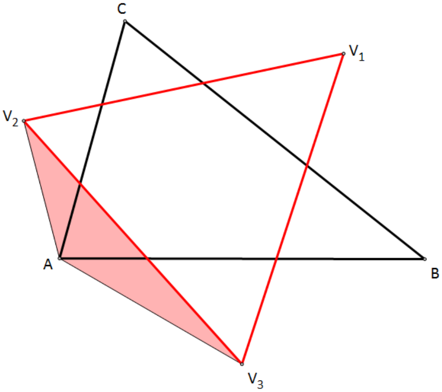
$$|\measuredangle V_2AV_3| = \alpha + 60^\circ$$

$$|AV_2| = \frac{b}{\sqrt{3}}$$

$$|AV_3| = \frac{c}{\sqrt{3}}$$

$$|V_2V_3|^2 = \frac{b^2}{3} + \frac{c^2}{3} - \frac{2bc}{3} \cdot \cos(\alpha + 60^\circ)$$

$\triangle V_1V_2V_3$ je jednakostraničan pa je $|V_1V_2| = |V_2V_3| = |V_1V_3| = v$



Neka su a, b, c duljine stranica, a α, β, γ veličine unutarnjih kutova $\triangle ABC$.

$$|\measuredangle V_2AV_3| = \alpha + 60^\circ$$

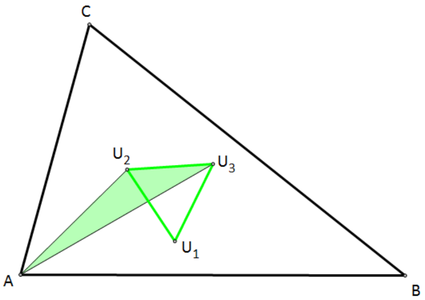
$$|AV_2| = \frac{b}{\sqrt{3}}$$

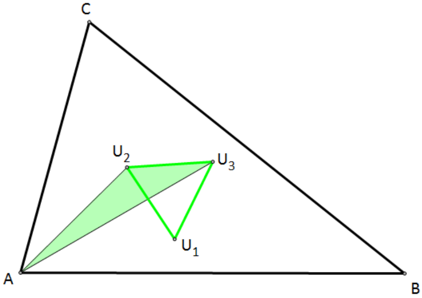
$$|AV_3| = \frac{c}{\sqrt{3}}$$

$$|V_2V_3|^2 = \frac{b^2}{3} + \frac{c^2}{3} - \frac{2bc}{3} \cdot \cos(\alpha + 60^\circ)$$

$\triangle V_1V_2V_3$ je jednakostraničan pa je $|V_1V_2| = |V_2V_3| = |V_1V_3| = v$

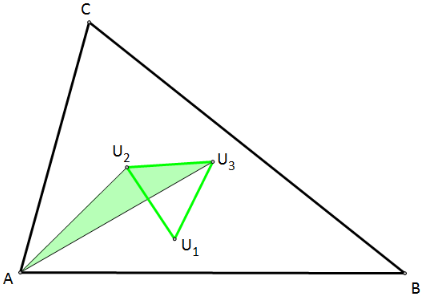
$$v^2 = \frac{b^2}{3} + \frac{c^2}{3} - \frac{2bc}{3} \cdot \cos(\alpha + 60^\circ) \tag{6}$$


C
U_2
U_3
U_1
A
B



$$|\measuredangle U_3AU_2| = \begin{cases} \alpha - 60^\circ & , \alpha \geq 60^\circ \\ 60^\circ - \alpha & , \alpha < 60^\circ \end{cases}$$

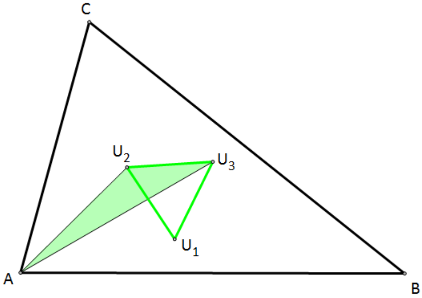
$$|\measuredangle U_3AU_2| = |\alpha - 60^\circ|$$



$$|\measuredangle U_3AU_2| = \begin{cases} \alpha - 60^\circ & , \alpha \geq 60^\circ \\ 60^\circ - \alpha & , \alpha < 60^\circ \end{cases}$$

$$|\measuredangle U_3AU_2| = |\alpha - 60^\circ|$$

$$|AU_2| = \frac{b}{\sqrt{3}}$$

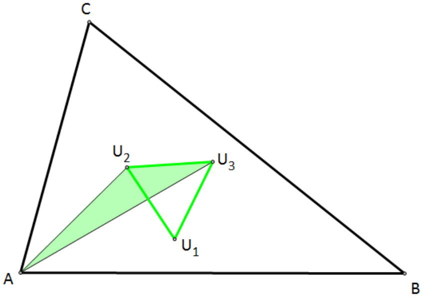


$$|\measuredangle U_3AU_2| = \begin{cases} \alpha - 60^\circ & , \alpha \geq 60^\circ \\ 60^\circ - \alpha & , \alpha < 60^\circ \end{cases}$$

$$|\measuredangle U_3AU_2| = |\alpha - 60^\circ|$$

$$|AU_2| = \frac{b}{\sqrt{3}}$$

$$|AU_3| = \frac{c}{\sqrt{3}}$$



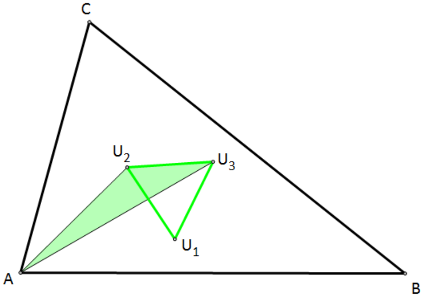
$$|\measuredangle U_3AU_2| = \begin{cases} \alpha - 60^\circ & , \alpha \geq 60^\circ \\ 60^\circ - \alpha & , \alpha < 60^\circ \end{cases}$$

$$|\measuredangle U_3AU_2| = |\alpha - 60^\circ|$$

$$|AU_2| = \frac{b}{\sqrt{3}}$$

$$|AU_3| = \frac{c}{\sqrt{3}}$$

$$|U_2U_3|^2 = \frac{b^2}{3} + \frac{c^2}{3} - \frac{2bc}{3} \cdot \cos|\alpha - 60^\circ|$$



$$|\measuredangle U_3AU_2| = \begin{cases} \alpha - 60^\circ & , \alpha \geq 60^\circ \\ 60^\circ - \alpha & , \alpha < 60^\circ \end{cases}$$

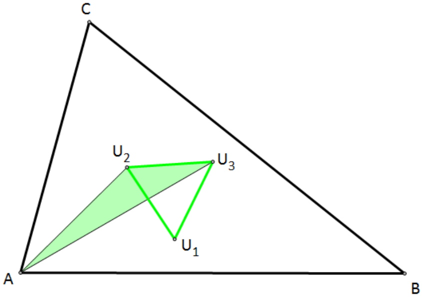
$$|\measuredangle U_3AU_2| = |\alpha - 60^\circ|$$

$$|AU_2| = \frac{b}{\sqrt{3}}$$

$$|AU_3| = \frac{c}{\sqrt{3}}$$

$$|U_2U_3|^2 = \frac{b^2}{3} + \frac{c^2}{3} - \frac{2bc}{3} \cdot \cos|\alpha - 60^\circ|$$

$\triangle U_1U_2U_3$ je jednakostraničan pa je $|U_1U_2| = |U_2U_3| = |U_1U_3| = u$



$$|\measuredangle U_3AU_2| = \begin{cases} \alpha - 60^\circ & , \alpha \geq 60^\circ \\ 60^\circ - \alpha & , \alpha < 60^\circ \end{cases}$$

$$|\measuredangle U_3AU_2| = |\alpha - 60^\circ|$$

$$|AU_2| = \frac{b}{\sqrt{3}}$$

$$|AU_3| = \frac{c}{\sqrt{3}}$$

$$|U_2U_3|^2 = \frac{b^2}{3} + \frac{c^2}{3} - \frac{2bc}{3} \cdot \cos|\alpha - 60^\circ|$$

$\triangle U_1U_2U_3$ je jednakostraničan pa je $|U_1U_2| = |U_2U_3| = |U_1U_3| = u$

$$u^2 = \frac{b^2}{3} + \frac{c^2}{3} - \frac{2bc}{3} \cdot \cos(\alpha - 60^\circ) \tag{7}$$

Iz (6) $$v^2 = \frac{b^2}{3} + \frac{c^2}{3} - \frac{2bc}{3} \cdot \cos(\alpha + 60^\circ)$$

i (7) $$u^2 = \frac{b^2}{3} + \frac{c^2}{3} - \frac{2bc}{3} \cdot \cos(\alpha - 60^\circ)$$ slijedi

$$v^2 - u^2 = \frac{2bc}{3}(\cos(\alpha - 60^\circ) - \cos(\alpha + 60^\circ))$$

Iz (6) $\quad v^2 = \frac{b^2}{3} + \frac{c^2}{3} - \frac{2bc}{3} \cdot \cos(\alpha + 60^\circ)$

i (7) $\quad u^2 = \frac{b^2}{3} + \frac{c^2}{3} - \frac{2bc}{3} \cdot \cos(\alpha - 60^\circ)$ $\quad$ slijedi

$$v^2 - u^2 = \frac{2bc}{3}(\cos(\alpha - 60^\circ) - \cos(\alpha + 60^\circ))$$

$$v^2 - u^2 = \frac{4}{\sqrt{3}} \cdot \frac{bc \sin \alpha}{2} = \frac{4}{\sqrt{3}} \cdot P_{\triangle ABC} \tag{8}$$

Iz (6) $v^2 = \frac{b^2}{3} + \frac{c^2}{3} - \frac{2bc}{3} \cdot \cos(\alpha + 60^\circ)$

i (7) $u^2 = \frac{b^2}{3} + \frac{c^2}{3} - \frac{2bc}{3} \cdot \cos(\alpha - 60^\circ)$ slijedi

$$v^2 - u^2 = \frac{2bc}{3}(\cos(\alpha - 60^\circ) - \cos(\alpha + 60^\circ))$$

$$v^2 - u^2 = \frac{4}{\sqrt{3}} \cdot \frac{bc \sin \alpha}{2} = \frac{4}{\sqrt{3}} \cdot P_{\triangle ABC} \tag{8}$$

$\triangle V_1 V_2 V_3$ i $\triangle U_1 U_2 U_3$ su jednakostranični pa je

$$P_{\triangle V_1 V_2 V_3} = P_v = \frac{v^2\sqrt{3}}{4}$$

Iz (6) $v^2 = \frac{b^2}{3} + \frac{c^2}{3} - \frac{2bc}{3} \cdot \cos(\alpha + 60^\circ)$

i (7) $u^2 = \frac{b^2}{3} + \frac{c^2}{3} - \frac{2bc}{3} \cdot \cos(\alpha - 60^\circ)$ slijedi

$$v^2 - u^2 = \frac{2bc}{3}(\cos(\alpha - 60^\circ) - \cos(\alpha + 60^\circ))$$

$$v^2 - u^2 = \frac{4}{\sqrt{3}} \cdot \frac{bc\sin\alpha}{2} = \frac{4}{\sqrt{3}} \cdot P_{\triangle ABC} \tag{8}$$

$\triangle V_1V_2V_3$ i $\triangle U_1U_2U_3$ su jednakostranični pa je

$$P_{\triangle V_1V_2V_3} = P_v = \frac{v^2\sqrt{3}}{4} \Rightarrow v^2 = \frac{4}{\sqrt{3}}P_v \tag{9}$$

Iz (6) $\quad v^2 = \frac{b^2}{3} + \frac{c^2}{3} - \frac{2bc}{3} \cdot \cos(\alpha + 60^\circ)$

i (7) $\quad u^2 = \frac{b^2}{3} + \frac{c^2}{3} - \frac{2bc}{3} \cdot \cos(\alpha - 60^\circ)$ slijedi

$$v^2 - u^2 = \frac{2bc}{3}(\cos(\alpha - 60^\circ) - \cos(\alpha + 60^\circ))$$

$$v^2 - u^2 = \frac{4}{\sqrt{3}} \cdot \frac{bc \sin \alpha}{2} = \frac{4}{\sqrt{3}} \cdot P_{\triangle ABC} \tag{8}$$

$\triangle V_1V_2V_3$ i $\triangle U_1U_2U_3$ su jednakostranični pa je

$$P_{\triangle V_1V_2V_3} = P_v = \frac{v^2\sqrt{3}}{4} \Rightarrow v^2 = \frac{4}{\sqrt{3}}P_v \tag{9}$$

$$P_{\triangle U_1U_2U_3} = P_u = \frac{u^2\sqrt{3}}{4}$$

Iz (6) $$v^2 = \frac{b^2}{3} + \frac{c^2}{3} - \frac{2bc}{3} \cdot \cos(\alpha + 60^\circ)$$
i (7) $$u^2 = \frac{b^2}{3} + \frac{c^2}{3} - \frac{2bc}{3} \cdot \cos(\alpha - 60^\circ)$$ slijedi

$$v^2 - u^2 = \frac{2bc}{3}(\cos(\alpha - 60^\circ) - \cos(\alpha + 60^\circ))$$

$$v^2 - u^2 = \frac{4}{\sqrt{3}} \cdot \frac{bc \sin\alpha}{2} = \frac{4}{\sqrt{3}} \cdot P_{\triangle ABC} \tag{8}$$

$\triangle V_1 V_2 V_3$ i $\triangle U_1 U_2 U_3$ su jednakostranični pa je

$$P_{\triangle V_1 V_2 V_3} = P_v = \frac{v^2\sqrt{3}}{4} \Rightarrow v^2 = \frac{4}{\sqrt{3}} P_v \tag{9}$$

$$P_{\triangle U_1 U_2 U_3} = P_u = \frac{u^2\sqrt{3}}{4} \Rightarrow u^2 = \frac{4}{\sqrt{3}} P_u \tag{10}$$

Iz (9) $$v^2 = \frac{4}{\sqrt{3}} P_v$$

i (10) $$u^2 = \frac{4}{\sqrt{3}} P_u$$ slijedi

$$v^2 - u^2 = \frac{4}{\sqrt{3}}(P_v - P_u)$$

Iz (9) $$v^2 = \frac{4}{\sqrt{3}} P_v$$

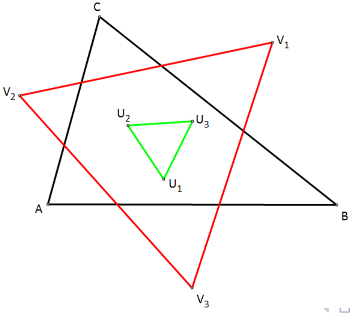
i (10) $$u^2 = \frac{4}{\sqrt{3}} P_u$$ slijedi

$$v^2 - u^2 = \frac{4}{\sqrt{3}}(P_v - P_u)$$

što sa (8) $$v^2 - u^2 = \frac{4}{\sqrt{3}} \cdot P_{\triangle ABC}$$ daje

$$\frac{4}{\sqrt{3}} \cdot P_{\triangle ABC} = \frac{4}{\sqrt{3}}(P_v - P_u)$$

$$P_{\triangle ABC} = P_v - P_u$$



Zadatak 1

- *Nacrtaj* $\triangle ABC$.

Zadatak 1

- *Nacrtaj* $\triangle ABC$.
- *Nad stranicama* $\triangle ABC$ *konstruiraj (s vanjske strane) jednakostranične trokute* $\triangle AC_1B$, $\triangle BA_1C$ *i* $\triangle CB_1A$.

Zadatak 1

- *Nacrtaj* $\triangle ABC$.
- *Nad stranicama* $\triangle ABC$ *konstruiraj (s vanjske strane) jednakostranične trokute* $\triangle AC_1B$, $\triangle BA_1C$ *i* $\triangle CB_1A$.
- *Trokutima* $\triangle AC_1B$, $\triangle BA_1C$ *i* $\triangle CB_1A$ *konstruiraj težišta* V_3, V_1, V_2.

Zadatak 1

- *Nacrtaj* $\triangle ABC$.
- *Nad stranicama* $\triangle ABC$ *konstruiraj (s vanjske strane) jednakostranične trokute* $\triangle AC_1B$, $\triangle BA_1C$ *i* $\triangle CB_1A$.
- *Trokutima* $\triangle AC_1B$, $\triangle BA_1C$ *i* $\triangle CB_1A$ *konstruiraj težišta* V_3, V_1, V_2.
- *Nacrtaj* $\triangle V_1V_2V_3$.
  - *Izmjeri duljine stranica* $\triangle V_1V_2V_3$.
  - *Izmjeri veličine unutarnjih kutova* $\triangle V_1V_2V_3$.

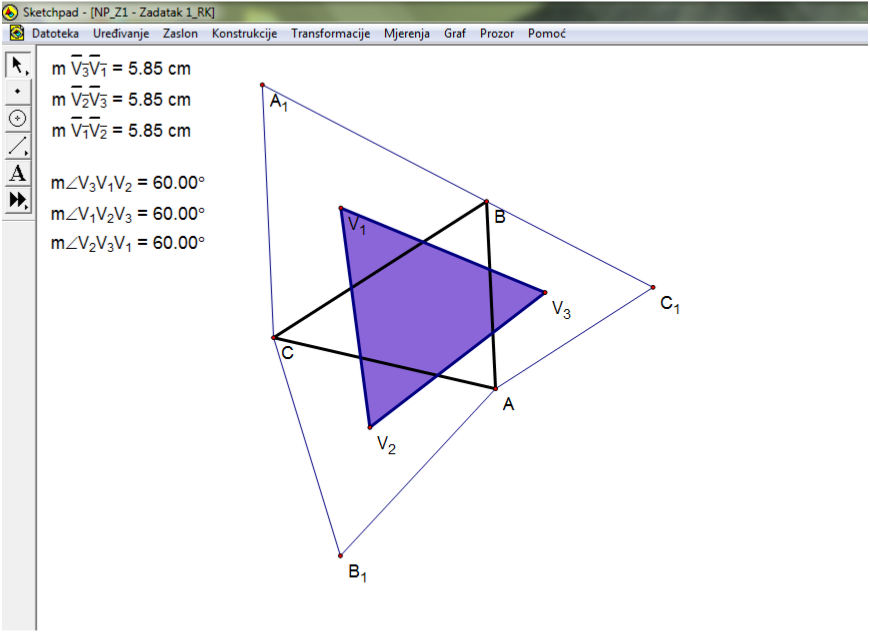
Zadatak 1

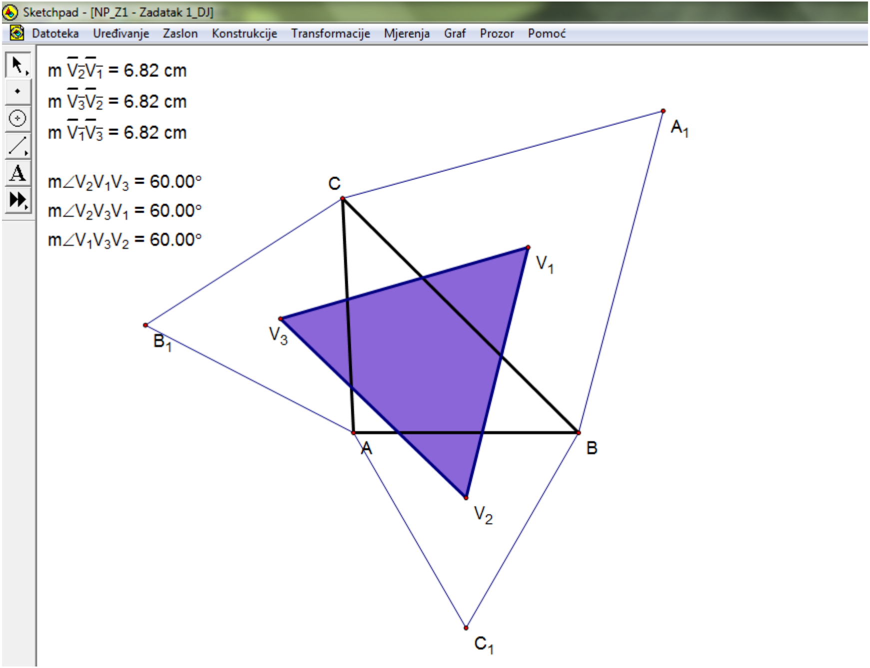
- *Nacrtaj* $\triangle ABC$.
- *Nad stranicama* $\triangle ABC$ *konstruiraj (s vanjske strane) jednakostranične trokute* $\triangle AC_1B$, $\triangle BA_1C$ *i* $\triangle CB_1A$.
- *Trokutima* $\triangle AC_1B$, $\triangle BA_1C$ *i* $\triangle CB_1A$ *konstruiraj težišta* V_3, V_1, V_2.
- *Nacrtaj* $\triangle V_1V_2V_3$.
  - *Izmjeri duljine stranica* $\triangle V_1V_2V_3$.
  - *Izmjeri veličine unutarnjih kutova* $\triangle V_1V_2V_3$.
- *Pomiči vrhove* $\triangle ABC$ *(pri tome promatraj* $\triangle V_1V_2V_3$*). Što zaključuješ o* $\triangle V_1V_2V_3$*?*

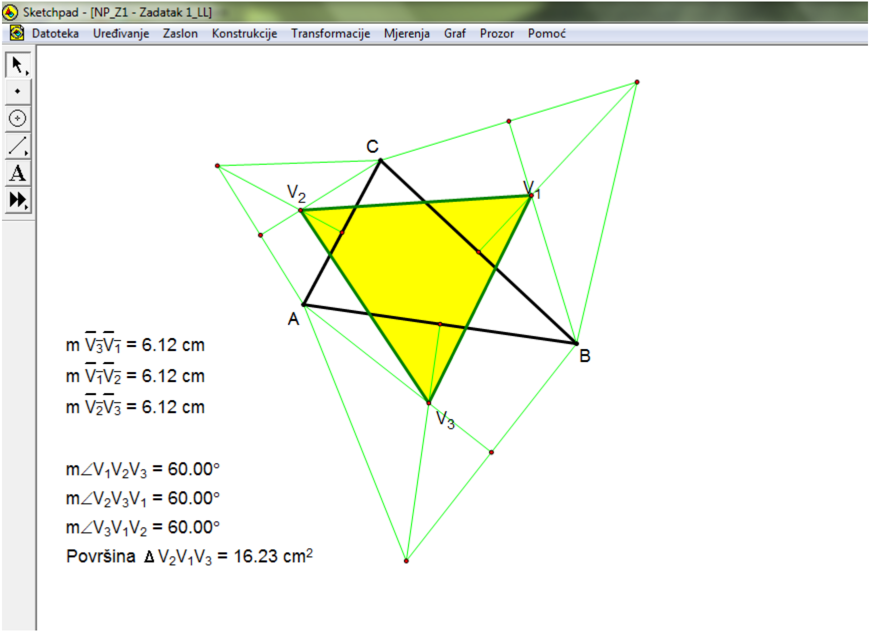
Zadatak 1

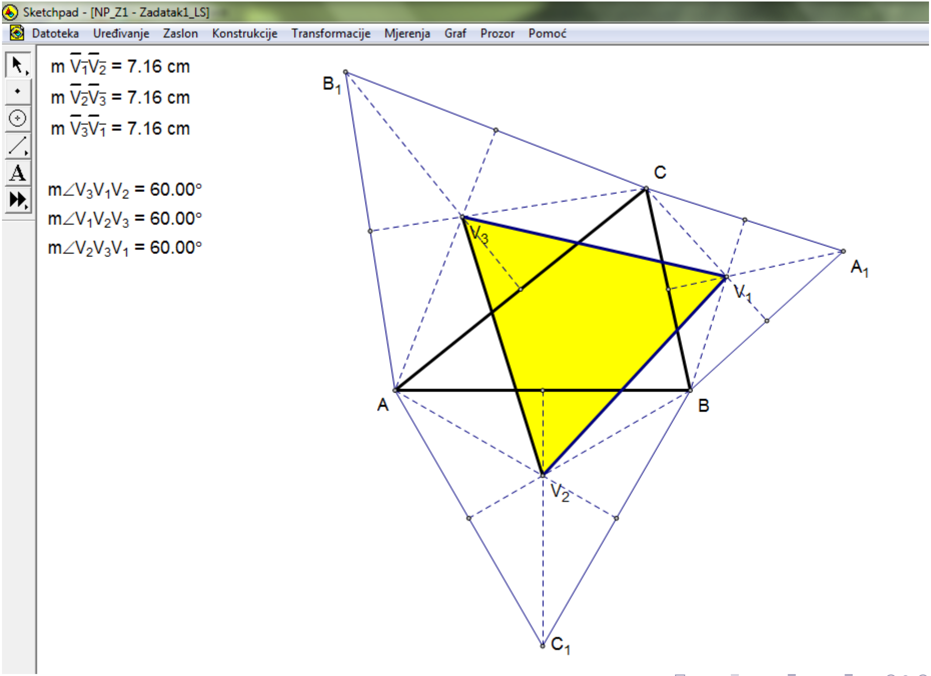
- *Nacrtaj* $\triangle ABC$.
- *Nad stranicama* $\triangle ABC$ *konstruiraj (s vanjske strane) jednakostranične trokute* $\triangle AC_1B$, $\triangle BA_1C$ *i* $\triangle CB_1A$.
- *Trokutima* $\triangle AC_1B$, $\triangle BA_1C$ *i* $\triangle CB_1A$ *konstruiraj težišta* V_3, V_1, V_2.
- *Nacrtaj* $\triangle V_1V_2V_3$.
  - *Izmjeri duljine stranica* $\triangle V_1V_2V_3$.
  - *Izmjeri veličine unutarnjih kutova* $\triangle V_1V_2V_3$.
- *Pomiči vrhove* $\triangle ABC$ *(pri tome promatraj* $\triangle V_1V_2V_3$*). Što zaključuješ o* $\triangle V_1V_2V_3$*?*
- *Trokut* $V_1V_2V_3$ *zove se* ***VANJSKI NAPOLEONOV TROKUT***.

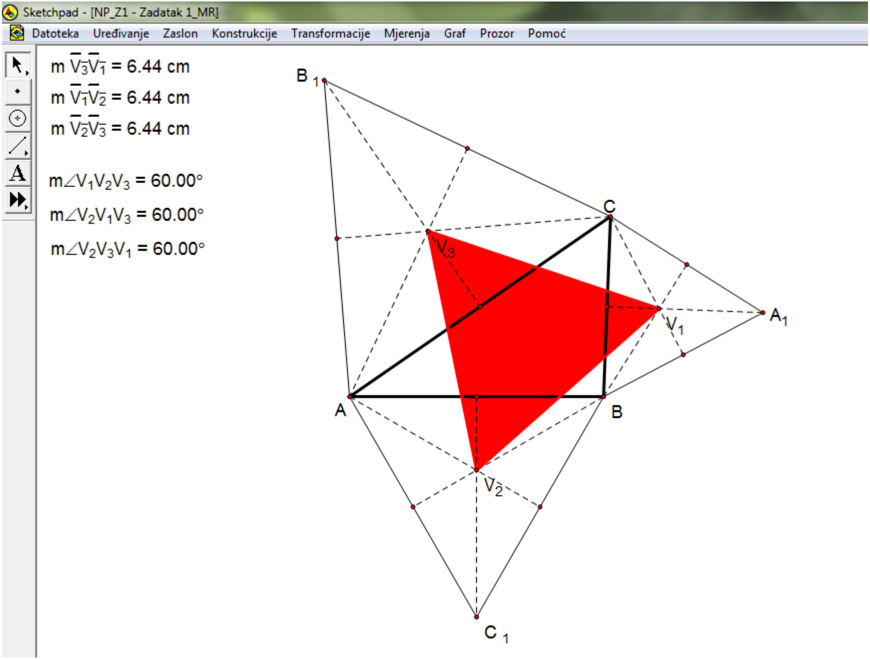
GSP

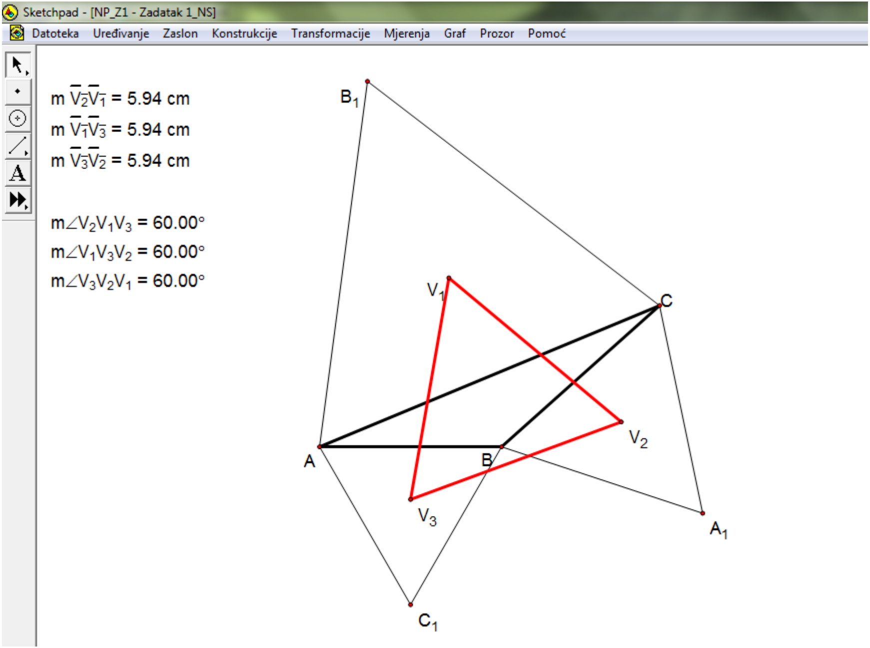

Sketchpad - [NP_Z1 - Zadatak 1_RK]
Datoteka Uređivanje Zaslon Konstrukcije Transformacije Mjerenja Graf Prozor Pomoć
m $\overline{V_3V_1}$ = 5.85 cm
m $\overline{V_2V_3}$ = 5.85 cm
m $\overline{V_1V_2}$ = 5.85 cm
m$\angle V_3V_1V_2$ = 60.00°
m$\angle V_1V_2V_3$ = 60.00°
m$\angle V_2V_3V_1$ = 60.00°
A_1
B
V_1
V_3
C_1
C
A
V_2
B_1


Sketchpad - [NP_Z1 - Zadatak 1_DJ]
Datoteka Uređivanje Zaslon Konstrukcije Transformacije Mjerenja Graf Prozor Pomoć
m $\overline{V_2}\overline{V_1}$ = 6.82 cm
m $\overline{V_3}\overline{V_2}$ = 6.82 cm
m $\overline{V_1}\overline{V_3}$ = 6.82 cm
m$\angle V_2V_1V_3$ = 60.00°
m$\angle V_2V_3V_1$ = 60.00°
m$\angle V_1V_3V_2$ = 60.00°
A_1
C
V_1
B_1
V_3
A
B
V_2
C_1


Sketchpad - [NP_Z1 - Zadatak 1_LL]
Datoteka Uređivanje Zaslon Konstrukcije Transformacije Mjerenja Graf Prozor Pomoć
C
V2
V1
A
B
V3
m V3V1 = 6.12 cm
m V1V2 = 6.12 cm
m V2V3 = 6.12 cm
m∠V1V2V3 = 60.00°
m∠V2V3V1 = 60.00°
m∠V3V1V2 = 60.00°
Površina ΔV2V1V3 = 16.23 cm2


Sketchpad - [NP_Z1 - Zadatak1_LS]
Datoteka Uređivanje Zaslon Konstrukcije Transformacije Mjerenja Graf Prozor Pomoć
m $\overline{V_1V_2}$ = 7.16 cm
m $\overline{V_2V_3}$ = 7.16 cm
m $\overline{V_3V_1}$ = 7.16 cm
m$\angle V_3V_1V_2$ = 60.00°
m$\angle V_1V_2V_3$ = 60.00°
m$\angle V_2V_3V_1$ = 60.00°
B_1
C
A_1
V_3
V_1
A
B
V_2
C_1

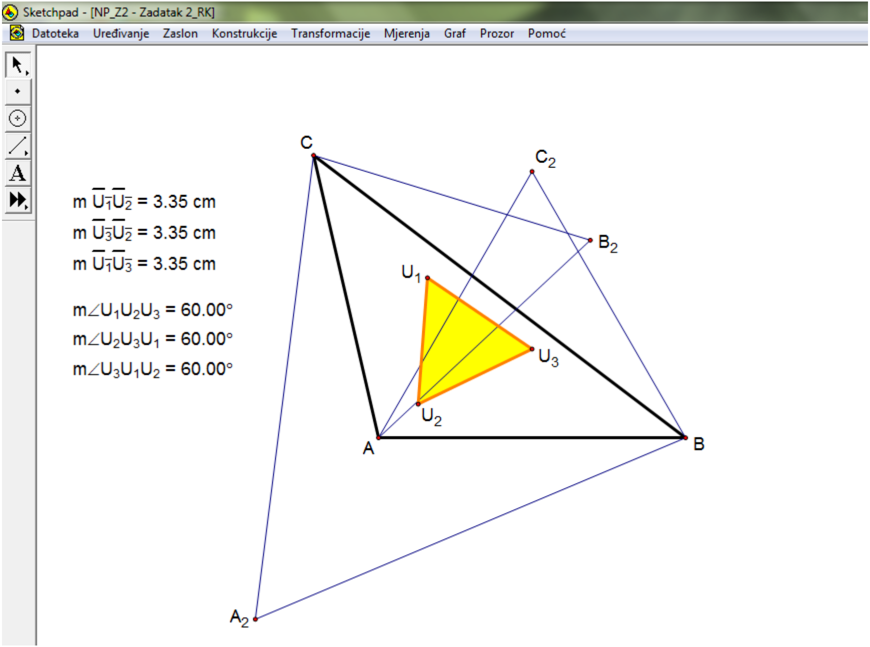

Sketchpad - [NP_Z1 - Zadatak 1_MR]
Datoteka Uređivanje Zaslon Konstrukcije Transformacije Mjerenja Graf Prozor Pomoć
m $\overline{V_3}\overline{V_1}$ = 6.44 cm
m $\overline{V_1}\overline{V_2}$ = 6.44 cm
m $\overline{V_2}\overline{V_3}$ = 6.44 cm
m$\angle V_1V_2V_3$ = 60.00°
m$\angle V_2V_1V_3$ = 60.00°
m$\angle V_2V_3V_1$ = 60.00°
B_1
C
V_3
V_1
A_1
A
B
V_2
C_1

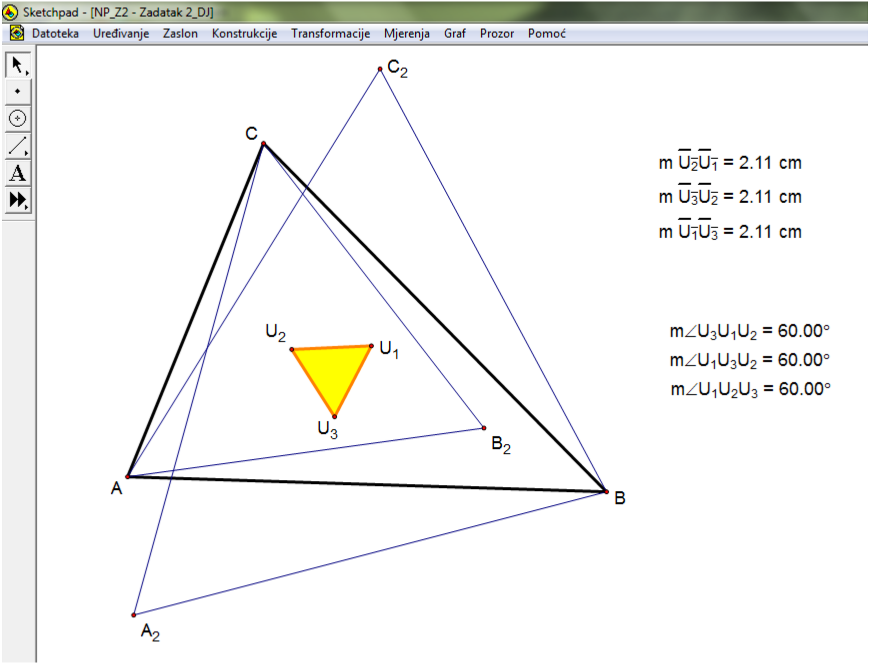

Sketchpad - [NP_Z1 - Zadatak 1_NS]
Datoteka Uređivanje Zaslon Konstrukcije Transformacije Mjerenja Graf Prozor Pomoć
m $\overline{V_2V_1}$ = 5.94 cm
m $\overline{V_1V_3}$ = 5.94 cm
m $\overline{V_3V_2}$ = 5.94 cm
m$\angle V_2V_1V_3$ = 60.00°
m$\angle V_1V_3V_2$ = 60.00°
m$\angle V_3V_2V_1$ = 60.00°
B_1
V_1
C
V_2
A
B
V_3
A_1
C_1

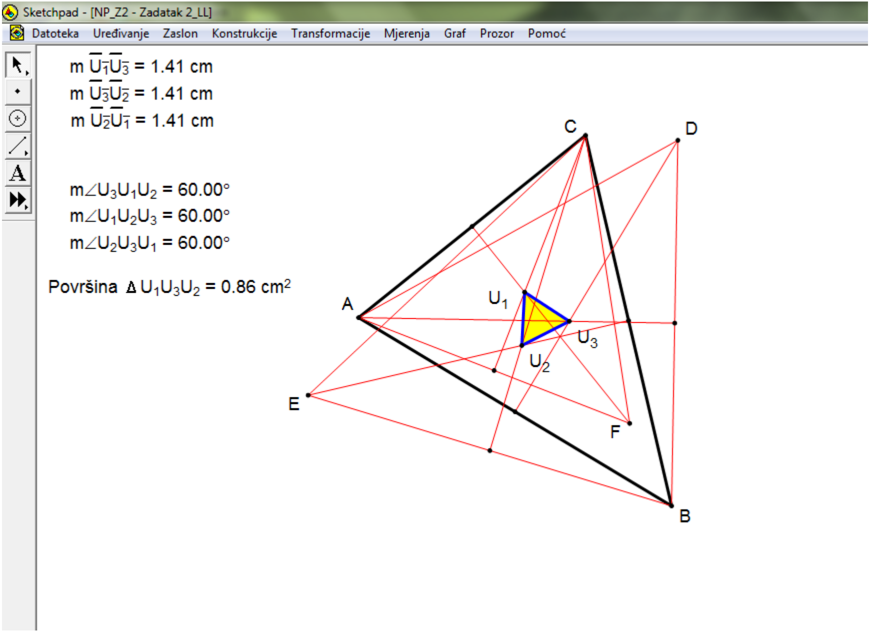
Zadatak 2

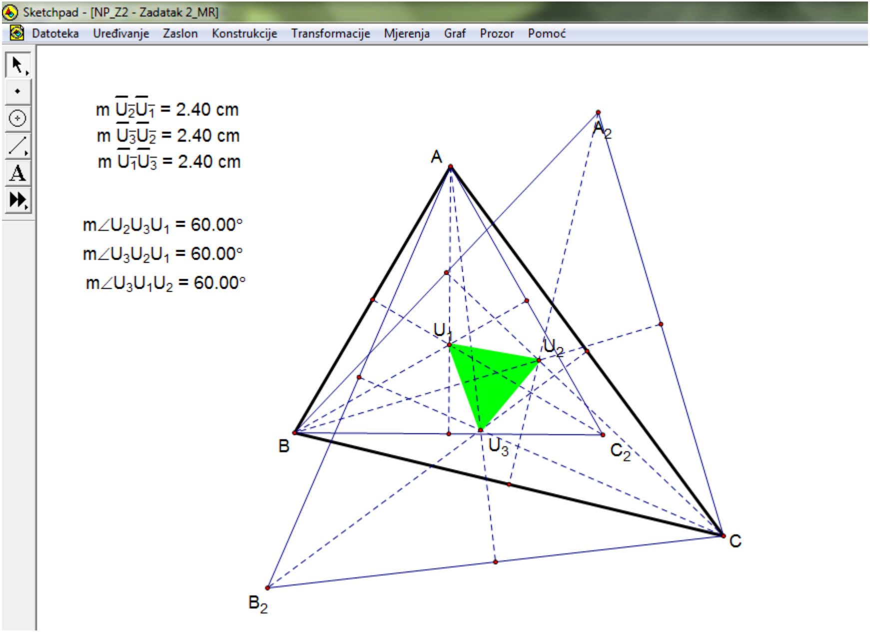
- *Nacrtaj* $\triangle ABC$.
- *Nad stranicama* $\triangle ABC$ *konstruiraj (s unutarnje strane) jednakostranične trokute* $\triangle ABC_2$, $\triangle BCA_2$ *i* $\triangle CAB_2$.
- *Trokutima* $\triangle ABC_2$, $\triangle BCA_2$ *i* $\triangle CAB_2$ *konstruiraj težišta* U_3, U_1, U_2.
- *Nacrtaj* $\triangle U_1 U_2 U_3$.
  - *Izmjeri duljine stranica* $\triangle U_1 U_2 U_3$.
  - *Izmjeri veličine unutarnjih kutova* $\triangle U_1 U_2 U_3$.
- *Pomiči vrhove* $\triangle ABC$ *(pri tome promatraj* $\triangle U_1 U_2 U_3$*). Što zaključuješ o* $\triangle U_1 U_2 U_3$*?*
- *Trokut* $U_1 U_2 U_3$ *zove se* ***UNUTARNJI NAPOLEONOV TROKUT***.

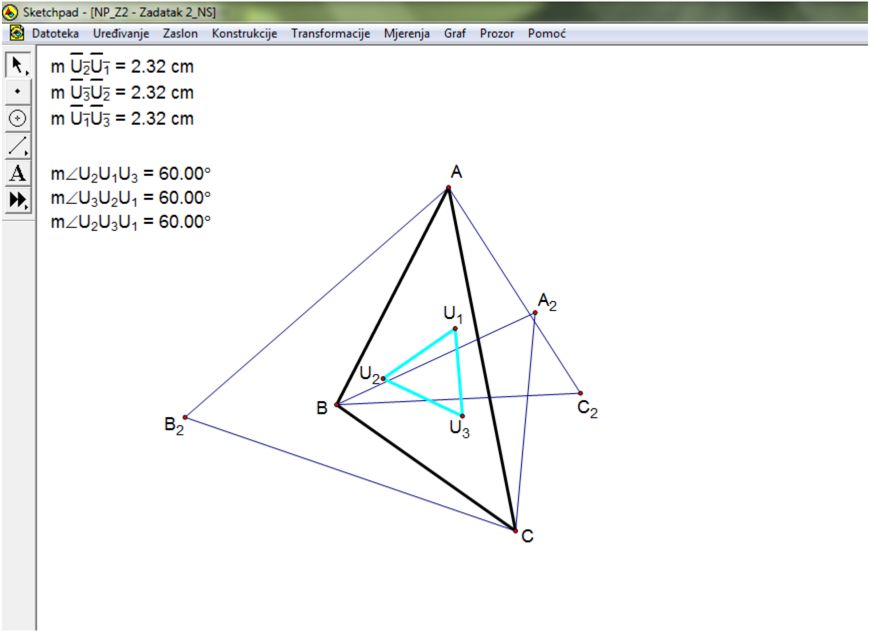
GSP


Sketchpad - [NP_Z2 - Zadatak 2_RK]
Datoteka Uređivanje Zaslon Konstrukcije Transformacije Mjerenja Graf Prozor Pomoć
$m\,\overline{U_1U_2} = 3.35$ cm
$m\,\overline{U_3U_2} = 3.35$ cm
$m\,\overline{U_1U_3} = 3.35$ cm
$m\angle U_1U_2U_3 = 60.00°$
$m\angle U_2U_3U_1 = 60.00°$
$m\angle U_3U_1U_2 = 60.00°$
C
C_2
B_2
U_1
U_3
U_2
A
B
A_2


Sketchpad - [NP_Z2 - Zadatak 2_DJ]
Datoteka Uređivanje Zaslon Konstrukcije Transformacije Mjerenja Graf Prozor Pomoć
C2
C
A
B
A2
B2
U1
U2
U3
m U2U1 = 2.11 cm
m U3U2 = 2.11 cm
m U1U3 = 2.11 cm
m∠U3U1U2 = 60.00°
m∠U1U3U2 = 60.00°
m∠U1U2U3 = 60.00°


Sketchpad - [NP_Z2 - Zadatak 2_LL]
Datoteka Uređivanje Zaslon Konstrukcije Transformacije Mjerenja Graf Prozor Pomoć
$m\ \overline{U_1}\overline{U_3} = 1.41$ cm
$m\ \overline{U_3}\overline{U_2} = 1.41$ cm
$m\ \overline{U_2}\overline{U_1} = 1.41$ cm
$m\angle U_3U_1U_2 = 60.00°$
$m\angle U_1U_2U_3 = 60.00°$
$m\angle U_2U_3U_1 = 60.00°$
Površina $\Delta\, U_1U_3U_2 = 0.86\ cm^2$
C
D
A
U_1
U_3
U_2
E
F
B


Sketchpad - [NP_Z2 - Zadatak 2_MR]
Datoteka Uređivanje Zaslon Konstrukcije Transformacije Mjerenja Graf Prozor Pomoć
m $\overline{U_2}\overline{U_1}$ = 2.40 cm
m $\overline{U_3}\overline{U_2}$ = 2.40 cm
m $\overline{U_1}\overline{U_3}$ = 2.40 cm
m∠$U_2U_3U_1$ = 60.00°
m∠$U_3U_2U_1$ = 60.00°
m∠$U_3U_1U_2$ = 60.00°
A_2
A
U_1
U_2
B
U_3
C_2
C
B_2


Sketchpad - [NP_Z2 - Zadatak 2_NS]
Datoteka Uređivanje Zaslon Konstrukcije Transformacije Mjerenja Graf Prozor Pomoć
m $\overline{U_2}\overline{U_1}$ = 2.32 cm
m $\overline{U_3}\overline{U_2}$ = 2.32 cm
m $\overline{U_1}\overline{U_3}$ = 2.32 cm
m$\angle U_2U_1U_3$ = 60.00°
m$\angle U_3U_2U_1$ = 60.00°
m$\angle U_2U_3U_1$ = 60.00°
A
A_2
U_1
U_2
B
C_2
B_2
U_3
C

Zadatak 3

- *Ponovi konstrukcije* $\triangle ABC$, $\triangle V_1V_2V_3$ *i* $\triangle U_1U_2U_3$ *iz prethodna dva zadatka.*

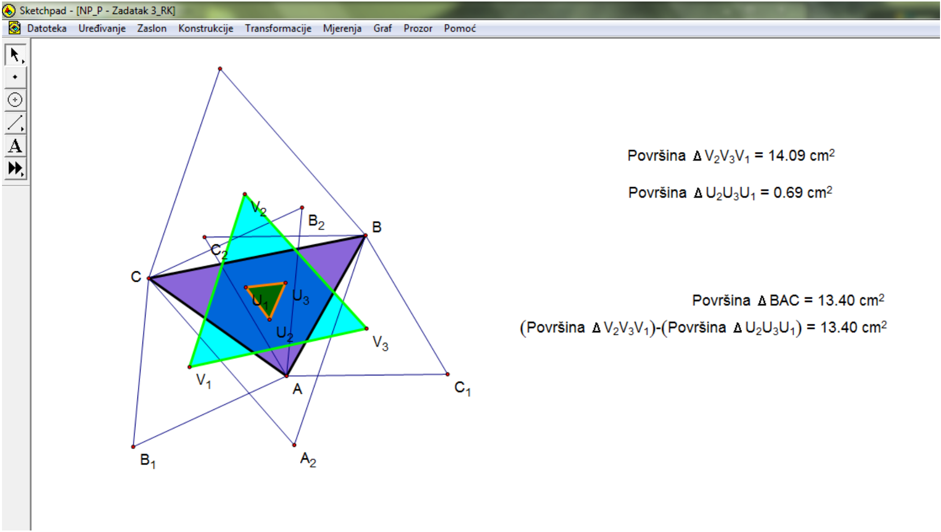
Zadatak 3

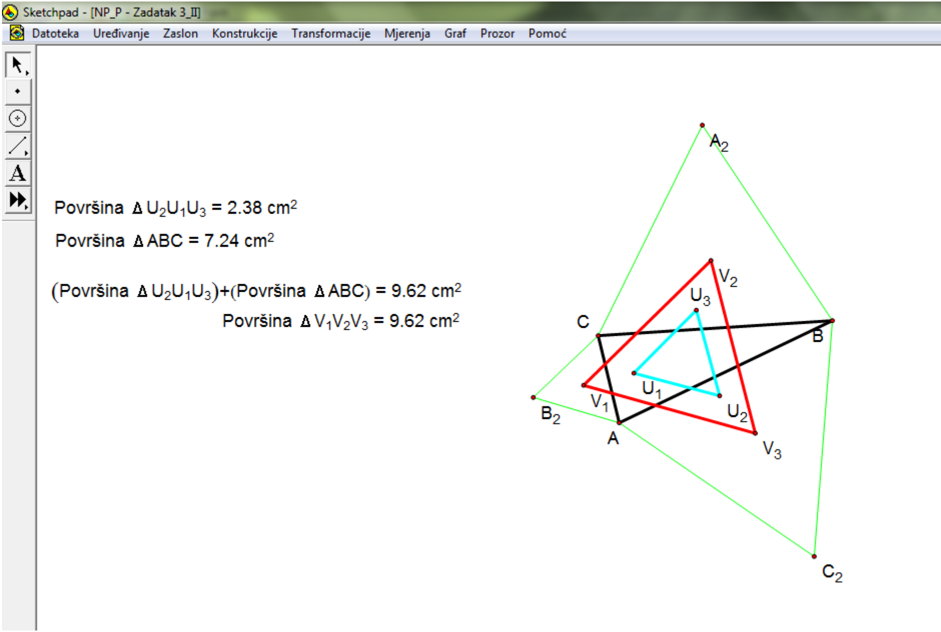
- *Ponovi konstrukcije* $\triangle ABC$, $\triangle V_1V_2V_3$ *i* $\triangle U_1U_2U_3$ *iz prethodna dva zadatka.*
- *Trokutima* $\triangle ABC$, $\triangle V_1V_2V_3$ *i* $\triangle U_1U_2U_3$ *izračunaj površine.*

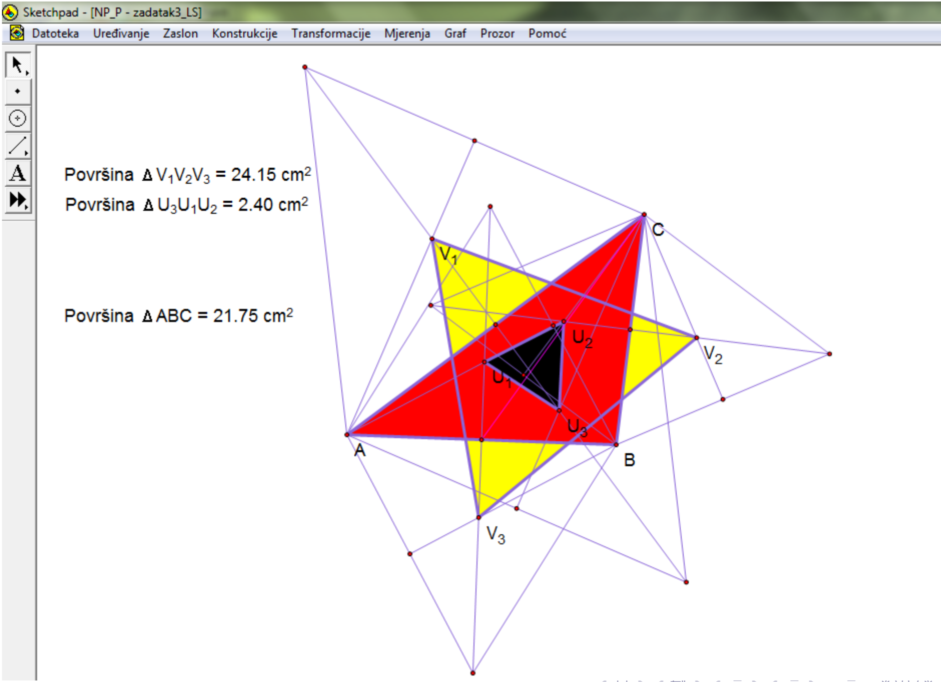
Zadatak 3

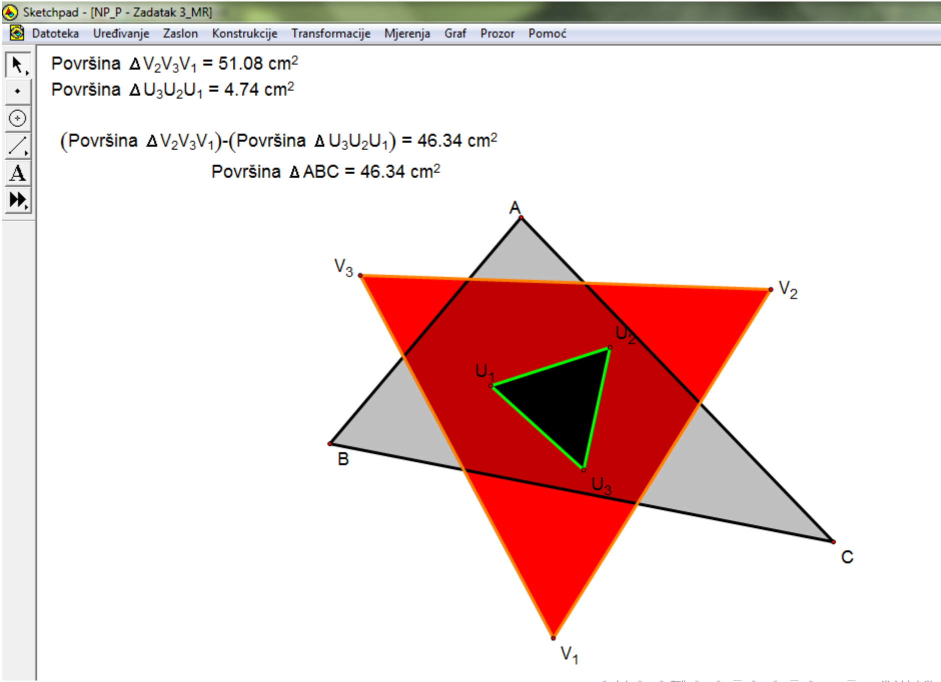
- *Ponovi konstrukcije* $\triangle ABC$, $\triangle V_1V_2V_3$ *i* $\triangle U_1U_2U_3$ *iz prethodna dva zadatka.*
- *Trokutima* $\triangle ABC$, $\triangle V_1V_2V_3$ *i* $\triangle U_1U_2U_3$ *izračunaj površine.*
- *Što zaključuješ?*

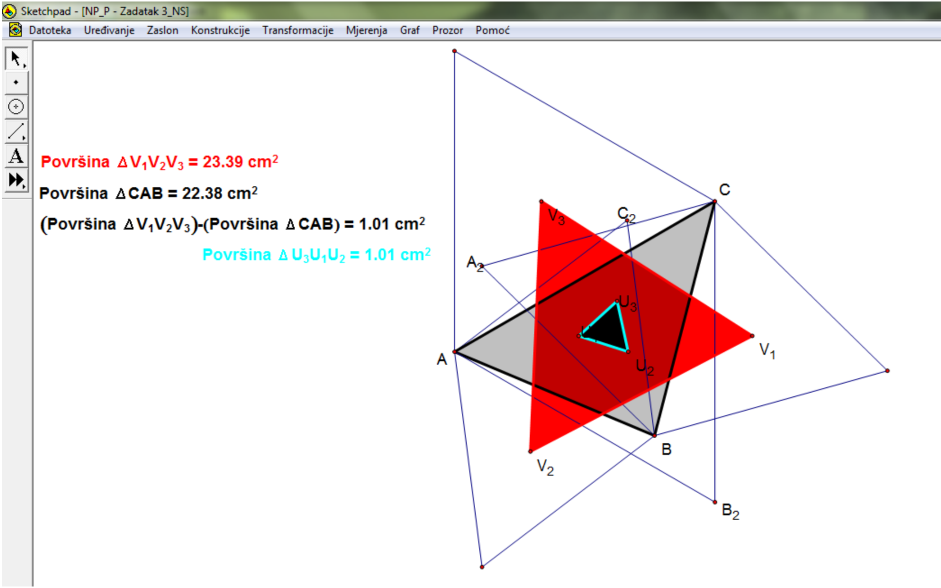
GSP


Sketchpad - [NP_P - Zadatak 3_RK]
Datoteka Uređivanje Zaslon Konstrukcije Transformacije Mjerenja Graf Prozor Pomoć
Površina Δ V2V3V1 = 14.09 cm2
Površina Δ U2U3U1 = 0.69 cm2
Površina Δ BAC = 13.40 cm2
(Površina Δ V2V3V1)-(Površina Δ U2U3U1) = 13.40 cm2
V2
B2
B
C2
C
U1
U3
U2
V3
V1
A
C1
B1
A2


Sketchpad - [NP_P - Zadatak 3_II]
Datoteka Uređivanje Zaslon Konstrukcije Transformacije Mjerenja Graf Prozor Pomoć
Površina $\Delta U_2U_1U_3$ = 2.38 cm^2
Površina ΔABC = 7.24 cm^2
(Površina $\Delta U_2U_1U_3$)+(Površina ΔABC) = 9.62 cm^2
Površina $\Delta V_1V_2V_3$ = 9.62 cm^2
A_2
V_2
U_3
C
B
U_1
V_1
B_2
U_2
A
V_3
C_2


Sketchpad - [NP_P - zadatak3_LS]
Datoteka Uređivanje Zaslon Konstrukcije Transformacije Mjerenja Graf Prozor Pomoć
Površina $\Delta V_1V_2V_3$ = 24.15 cm^2
Površina $\Delta U_3U_1U_2$ = 2.40 cm^2
Površina ΔABC = 21.75 cm^2
V_1
C
U_2
V_2
U_1
U_3
A
B
V_3


Sketchpad - [NP_P - Zadatak 3_MR]
Datoteka Uređivanje Zaslon Konstrukcije Transformacije Mjerenja Graf Prozor Pomoć
Površina $\Delta V_2V_3V_1$ = 51.08 cm^2
Površina $\Delta U_3U_2U_1$ = 4.74 cm^2
(Površina $\Delta V_2V_3V_1$)-(Površina $\Delta U_3U_2U_1$) = 46.34 cm^2
Površina ΔABC = 46.34 cm^2
A
V_3
V_2
U_2
U_1
B
U_3
C
V_1



Sketchpad - [NP_P - Zadatak 3_NS]
Datoteka Uređivanje Zaslon Konstrukcije Transformacije Mjerenja Graf Prozor Pomoć
Površina $\Delta V_1V_2V_3$ = 23.39 cm^2
Površina ΔCAB = 22.38 cm^2
(Površina $\Delta V_1V_2V_3$)-(Površina ΔCAB) = 1.01 cm^2
Površina $\Delta U_3U_1U_2$ = 1.01 cm^2
C
C_2
V_3
A_2
U_3
A
V_1
U_2
B
V_2
B_2

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Hvala na pažnji!

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